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A TWO-STEP INERTIAL METHOD WITH A NEW STEP-SIZE RULE FOR QUASIMONOTONE VARIATIONAL INEQUALITIES IN HILBERT SPACES

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Dedicated to Professor Hari Mohan Srivastava on the Occasion of His 85th Birthday

Abstract. In this paper, a two-step inertial Tseng extragradient method involving self-adaptive and Armijo-like step sizes is introduced for solving variational inequalities with a quasimonotone and Lipschitz cost function in the setting of a real Hilbert space. Weak convergence of the sequence generated by the proposed algorithm is proved. An interesting feature of the proposed algorithm is its ability to select the better step size between the self-adaptive and Armijo-like options at each iteration step. Finally, the algorithm accelerates and complements several existing iterative algorithms for solving variational inequalities in Hilbert spaces.

Keywords. Variational inequality, Quasimonotone, Armijo-like step-size, Self-adaptive, Weak convergence, Hilbert spaces.

1. Introduction

Let H be a real Hilbert space with the inner product$\langle\cdot,\cdot\rangle$ and the induce norm$\|\cdot\|$. Then we suppose that C is nonempty, closed and convex subset of H ($C \subset H$). In this work, we focus on the following problem:

$$\text{Find } \; x^* \in C \; \text{ such that } \; \langle Ax^*,\, x - x^*\rangle \geq 0,\; \forall x \in C. \tag{1.1}$$

Problem (1.1) is popularly known as the variational inequality problem (VIP) ($VI(C,A)$, for short). Let Ω denote the set of solutions of $VI(A,C)$ (1.1). That is, $\Omega := \{q \in C : \langle A(q), x - q \geq 0\rangle\}$. The VIP has been used as a modeling tool for the study of various real-life problems such as the obstacle problem, contact problem, traffic network problem, and optimal control problem, and have also been applied in diverse areas of study such as optimization theory, nonlinear analysis, and computational mechanics. Those applications of VI(A, C) (1.1) mentioned about are discussed in [8, 9, 15, 17, 30].

Numerous iterative algorithms for solving $VI(A,C)$ (1.1) have been extensively studied and developed by many authors (see, for example [2, 6, 18, 37] and the references therein). The fundamental idea involves extending the projected gradient method, originally designed to solve a constrained minimization problem involving f over some nonempty closed and convex set say, C. The iterative procedure is given by:

$$\begin{cases} x_1 \in \mathbb{R}^n, \\ x_{k+1} = P_C\left(x_k - \alpha_k \nabla f(x_k)\right), \end{cases} \tag{1.2}$$

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where P_C is the metric projection onto C, α_k is a positive real sequence that satisfies some certain conditions and ∇f is the gradient of the smooth function f [36]. One of the early generalizations of the projected gradient method to the setting of VIP was the method proposed by Goldstein [31]:

$$\begin{cases} x_1 \in \mathbb{R}^n \\ x_{k+1} = P_C(x_k - \lambda A x_k), \end{cases} \tag{1.3}$$

where λ is a positive real number, P_C is the projection operator and A is a given mapping. But the convergence of this method is only guaranteed under a highly restrictive condition that the operators are strongly monotone or inverse strongly monotone (see for [35]).

Korpelevich [16] improved Goldstein's method by extending the operator A to monotone and L-Lipschitz continuous mapping. He then introduced the famous extragradient method (EGM) given by:

$$\begin{cases} x_1 \in \mathbb{R}^n, \\ y_k = P_C(x_k - \lambda A x_k), \\ x_{k+1} = P_C(x_k - \lambda A y_k), \end{cases} \tag{1.4}$$

where $\lambda \in (0, 1/L)$.

Remark 1.1. It is well-known that one of the drawbacks of the EGM arises from its requirement to perform two projections onto the closed convex set during each iteration. This requirement can be computationally expensive especially when the structure of C is not simple. Another drawback of this method is the fact that parameter λ depends on the explicit value of the Lipschitz constant L which is can be challenging to obtain.

Many authors have tried to address Remark 1.1 in different directions (see [5, 7, 10, 12, 22, 28, 29, 38]). Censor [4] introduced a new method called the subgradient extragradient method (SEGM) which is defined as follows:

$$\begin{cases} x_1 \in H, \\ y_k = P_C\left(x_k - \lambda A x_k\right), \\ T_n = \{w \in H | \left\langle x_k - \lambda A x_k - y_k, w - y_k \right\rangle \leq 0\}, \\ x_{k+1} = P_{T_k}(x_k - \lambda A y_k), k > 1, \end{cases}$$

Another modification of the EGM was the method introduced by Tseng [34]. His idea was to replace the second projection onto C by a function evaluation. His algorithm is the following:

$$\begin{cases} x_1 \in H, \\ y_k = P_C\left(x_k - \lambda A x_k\right), \\ x_{k+1} = y_k - \lambda\left(A y_k - A x_k\right), \end{cases} \tag{1.5}$$

where $\lambda \in (0, 1/L)$. Tseng [34] proved that the sequence generated by [34] converges weakly to a point in Ω. The advantage of the above Tseng's method is that it requires only one computation of projection onto the feasible set C and two evaluations of A per iteration.

It is worthy of mention that the sequences generated by all the algorithms above may have slow convergence properties. To accelerate the convergence, several authors have adopted the inertial acceleration technique which dates back to the early work of Polyak [25] in the setting of convex minimization. Alvarez and Attouch [1] adopted this principle and extended it to general maximal monotone operators through a proximal point framework. They proposed a new algorithm called inertial proximal point

algorithm which is defined as:

$$\begin{cases} x_0, x_1 \in H, \\ y_k = x_k + \alpha_k(x_k - x_{k-1}), \\ x_{k+1} = (I + \lambda_k A)^{-1} y_k, \end{cases} \tag{1.6}$$

where $(I + \lambda_k A)^{-1}$ is the resolvent of the maximal monotone operator A and λ_k is a positive sequence that satisfies some appropriate conditions. Then, the authors proved that the sequence x_k generated by (1.6) converges weakly to a zero of A provided $\alpha_n \in [0, 1)$ satisfies the following condition:

$$\sum_{k=1}^{\infty} \alpha_k \|x_k - x_{k-1}\|^2 < +\infty. \tag{1.7}$$

In the setting of VIP, Thong and Hieu [33] introduced an inertial step in the Tseng's method for a better performance in a real Hilbert space. Their algorithm is defined as:

$$\begin{cases} x_0, x_1 \in H, \\ w_k = x_k + \alpha_k(x_k + x_{k+1}), \\ y_k = P_C(w_k - \lambda_k w_k), \\ x_{k+1} = y_k - \lambda_k(Ay_k - Aw_k), \end{cases} \tag{1.8}$$

where A is monotone and Lipschitz continuous and λ_k is a step-size obtained using Armijo-like step size rule.

Some results from using one-step inertial have shown that algorithms with this acceleration technique may fail to outperform their counterpart that does not involve this step. A counter example was given in [27] which which one-step inertial extrapolation fails to provide acceleration. Polyak mentioned in [26] that the use of inertial of more than two points x_k, x_{k-1} could provide acceleration. Polyak [26] also discussed that the multi-step inertial methods can boost the speed of optimization methods though neither the convergence nor the rate of such multi-step inertial methods was established in [26]. Recently, several authors have explore the concept of two-step inertia to accelerate convergence (see for example, [3, 13, 24]).

Remark 1.2. It is important to note that all the improvements of the EGM and their accelerated versions mentioned above have not fully addressed Remark 1.1. They are yet to dispense with the dependency of the step-size on the explicit value of the Lipschitz constant.

In recent years, different rules of selecting the step-size have been discussed since the arising of stochastic approximation methods. Liu and Yang [20] introduced a new self-adaptive method for solving variational inequalities with Lipschitz continuous and quasimonotone mapping (or Lipschitz continuous mapping without monotonicity) in real Hilbert space.The method is defined as:

$$\begin{cases} x_1 \in H, \\ y_k = P_C\left(x_k - \lambda_k A x_k\right), \\ \lambda_{k+1} = \begin{cases} \min\left\{\frac{\mu\|x_k - y_k\|}{\|Ax_k - Ay_k\|}, \lambda_k\right\}, & \text{if } Ax_k - Ay_k \neq 0, \\ \lambda_k, & \text{otherwise}, \end{cases} \\ x_{k+1} = y_k - \lambda_k\left(Ay_k - Ax_k\right). \end{cases}$$

They introduced the adaptive step-size to modify the gradient method and get the weak convergence without knowing the Lipschitz constant. And another popular step-size is Armijo like step-size, which is a fundamental step size selection technique in optimization algorithms. It ensures sufficient decrease in the objective function while balancing computational efficiency.

Recently Mewomo et al. [23] used the two-step inertial acceleration strategy and the adaptive step-size to introduce a two-step inertial Tseng method involving quasimonotone and uniformly continuous operator. Their algorithm is given by:

Algorithm: 1

Initialization: *Given* $\gamma > 0, l \in (0,1), \mu \in (0,1)$. *Let* $x_{-1}, x_0, x_1 \in \mathcal{H}$ *be arbitrary points and given* x_{k-2}, x_{k-1}, x_k

Iterative: Calculate x_{k+1} **as follows**

Step 1: Set $w_k = x_k + \alpha(x_k - x_{k-1}) + \beta(x_{k-1} - x_{k-2})$ and compute

$$y_k = P_C\left(w_k - \lambda_k A w_k\right),$$

where $\lambda_k^{(2)} = \gamma l^{m_k}$ *and* m_k *is the smallest nonnegative integer m such that*

$$\lambda_k \left\|Aw_k - Ay_k\right\| \leq \mu \left\|w_k - y_k\right\|. \tag{1.9}$$

If $y_k = w_k$ *then stop:* y_k *is a solution of the problem (VIP). Otherwise,*

Step 2: Compute

$$x_{k+1} = y_k - \lambda_k(Ay_k - w_k).$$

Set $k := k + 1$ *and go to* ***Step 1****.*

They proved weak convergence of the sequence generated by their proposed algorithm.

From different angle, Peng et al. [14] used the Armijo-like condition to proposed a modified Tseng method for solving pseudo-monotone variational inequality problems and the fixed point problems of the demi-contractive mappings. Their algorithm is defined as:

Algorithm: 2

Initialization: *Given* $k = 1, \rho > 0, \lambda_0 > 0, \gamma > 0, l \in (0,1), \mu \in (0,1)$. *Let* $x_0, x_1 \in \mathcal{H}$ *be arbitrary points*

Step 1: Given the current iterate x_{k-1}, x_k

$$w_k = x_k + \alpha_k(x_k - x_{k-1}),$$

where

$$\alpha_k = \begin{cases} \min\left\{\frac{\xi_k}{\|x_k - x_{k-1}\|}, \rho\right\}, & \text{if} x_k \neq x_{k-1}, \\ \rho, & \text{otherwise} \end{cases}$$

Step 2: Compute

$$y_k = P_C\left(w_k - \lambda_k A w_k\right),$$

where $\lambda_k = min(\lambda_k^{(1)}, \lambda_k^{(2)})$

$$\lambda_k^{(1)} = \begin{cases} \min\left\{\frac{\mu(\|w_k - y_k\|)}{A(w_k) - A(y_k)}, \lambda_{k-1}\right\}, & \text{if } A\left(w_k\right) - A\left(y_k\right) \neq 0, \\ \lambda_{k-1}, & \text{otherwise .} \end{cases} \tag{1.10}$$

where $\lambda_k^{(2)} = \gamma l^{m_k}$ *and* m_k *is the smallest nonnegative integer m such that*

$$\lambda_k \left\|Aw_k - Ay_k\right\| \leq \mu \left\|w_k - y_k\right\|. \tag{1.11}$$

$$q_k = (1 - \beta_k)z_k + \beta_k U z_k, \tag{1.12}$$

where $z_k = y_k + \lambda_k(Aw_k - Ay_k)$.

If $w_k = y_k = q_k$, then stop w_k is a solution of $w_k \in VI(C, A) \cap Fix(U)$. Otherwise

Step 3: Compute

$$x_{k+1} = \eta_k D_k + (1 - \eta_k) q_k$$

where $D_k = (1-\theta_k)f(x_k) + \theta_k f(x_{k-1})$, *Set* $k := k+1$ *and go to* ***Step 1***.

In the step2 of Algorithm 2, Peng et al choose the step-size as the minimum of the adaptive step-size and the Armijo rule step size. In this paper, inspired by the work of [14] and [23], we have the following contributions:

- We propose a novel algorithm that incorporates a two-step inertial technique to accelerate convergence. Unlike traditional one-step inertial methods, our approach leverages historical information from two preceding iterations, enhancing the algorithm's momentum and convergence speed.
- Our work extends the applicability of Tseng's extragradient method to quasimonotone VIPs, a broader and more general class of problems compared to the monotone or strongly monotone cases typically studied. This extension is significant because quasimonotonicity covers a wider range of practical problems while requiring weaker assumptions for convergence.

2. Preliminaries

In this section, we review the definitions and lemmas required for this article.

Definition 2.1. Let H be a real Hilbert space. An operator $A : H \to H$ is said to be:
(i) *Lipschitz continuous* on H, if there exist a constant $L > 0$ such that

$$||Ax - Ay|| \leq L||x - y||, \forall x, y \in H.$$

(ii) uniformly continuous, if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, such that

$$||Ax - Ay|| < \epsilon, \text{ whenever } ||x - y|| < \delta, \ \forall x, y \in H;$$

(iii) sequentially weakly-strongly continuous, if for each sequence x_k, we have $x_k \rightharpoonup x \in H$ implies that $Ax_k \to Ax \in H$;
(iv) sequentially weakly continuous, if for each sequence x_n, we have $x_n \rightharpoonup x \in H$ implies that $Ax_k \to Ax \in H$;
(v) monotone, if $\langle Ax - Ay, x - y\rangle \geq 0, \forall x, y \in H$;
(vi) α-strongly pseudomonotone, if there exists $\alpha > 0$ such that

$$\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, y - x\rangle \geq \alpha||x - y||^2, \ \forall x, y \in H;$$

(vii) pseudomonotone, if

$$\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, y - x\rangle \geq 0, \ \forall x, y \in H;$$

(viii) quasimonotone, if

$$\langle Ax, y - x\rangle > 0 \Rightarrow \langle Ay, y - x\rangle \geq 0, \ \forall x, y \in H;$$

From the definition above, the following implications hold: $(v) \Longrightarrow (vii) \Longrightarrow (viii)$ but the converse is not true in general. We also note that uniform continuity is a weaker notion than Lipschitz continuity.

Furthermore, it is well known that if D is a convex subset of H, then $A : D \to H$ is uniformly continuous if and only if, for every $\epsilon > 0$, there exists a constant $K < +\infty$ such that

$$||Ax - Ay|| \leq K||x - y|| + \epsilon, \ \forall x, y \in D.$$

Let S_D be the solution set of the dual formulation of the VIP (1.1) defined as: find $x^* \in C$ such that

$$\langle Az, \ z - x^*\rangle \geq 0, \ \forall z \in C.$$

Then, S_D is a closed and convex subset of C, and since A is continuous and C is convex, we have that $S_D \subset S$. We have the following result on the solution set of the dual VIP .

Lemma 2.2. Let $x, y, z \in H$ and $a, b \in \mathbb{R}$. Then

$$\begin{aligned}||(1+a)x-(a-b)y-bz||^2 =&(1+a)||x||^2-(a-b)||y||^2-b||z||^2\\ &+(1+a)(a-b)||x-y||\\ &+b(1+a)||x-z||^2-b(a-b)||y-z||^2.\end{aligned}$$

Lemma 2.3. Let C be a nonempty closed subset of a real Hilbert space H. $\forall v \in H$ and $\forall z \in C$, we have

$$z = P_C v \Leftrightarrow \langle v - z, z - y\rangle \geq 0, \forall y \in C.$$

Lemma 2.4. The following identities hold for all $u, v \in H$:

$$2\langle u, v\rangle = \|u\|^2 + \|v\|^2 - \|u - v\|^2 = \|u + v\|^2 - \|u\|^2 - \|v\|^2.$$

Lemma 2.5. ([19]) Suppose either
(a) A is pseudomonotone on C and $S = \emptyset$;
(b) A is the gradient of G, where G is a differential quasiconvex function on an open set $K \supset C$ and attains its global minimum on C;
(c) A is quasimonotone on C, $A \neq 0$ on C and C is bounded;
(d) A is quasimonotone on C, $A = 0$ on C and there exists a positive number r such that, for every $x \in C$ with $||x|| \geq r$, there exists $y \in C$ such that $||y|| \leq r$ and $\langle Ax, y - x\rangle$;
(e) A is quasimonotone on C, $intC = \emptyset$ and there exists $x^* \in S$ such that $Ax^* \neq 0$,

then S_D is nonempty.

Lemma 2.6. ([11]) Let H be a Hilbert space and $A : H \to H$ be a uniformly continuous operator. Suppose $x \in H$ and $\psi \geq \sigma > 0$. The following inequality holds:

$$\frac{\|x - P_C(x - \psi Ax)\|}{\psi} \leq \frac{\|x - P_C(x - \sigma Ax)\|}{\sigma}.$$

3. Propose the Algorithm

The subsequent section outlines the presentation of the algorithm. To ensure the weak convergence properties of the algorithm are valid, we propose the following assumptions.
Assumption 1.
(i) $S_D \neq \emptyset$;
(ii) A is uniformly continuous on H;
(iii) A satisfies the following condition: whenever $x_k \subset C$ and $x_k \rightharpoonup v^*$, one has $\|Av^*\| \leq \liminf_{n\to\infty} \|Ax_k\|$;
(iv) A is quasimonotone on H;
(v) The set $\{v \in C : Av = 0\} \backslash S_D$ is a finite set.

Assumption 2. Assume that α and β meet the following conditions:
(a) $0 \leq \alpha \leq \frac{1-\mu}{3+\mu}$;
(b) $max\{2\alpha(\frac{1-\mu}{3+\mu}) - (1-\alpha), \frac{1}{2}[\alpha(1+\mu) - (\frac{(1-\mu)(1-\alpha)^2}{1+\alpha})]\} < \beta \leq 0$;
(c) $2\alpha^2\mu - (1-3\alpha) + \mu(1-\alpha) - \beta(4\alpha + 3 - \mu) + 2\mu\beta^2 < 0$.

Under the assumptions above, we propose an algorithm for solving VIP (1.1):

Algorithm: 3

Initialization: *Given* $\gamma > 0, l \in (0,1), \mu \in (0,1)$. *Let* $x_{-1}, x_0, x_1 \in \mathcal{H}$ *be arbitrary points and given* x_{k-2}, x_{k-1}, x_k

Iterative: Calculate x_{k+1} as follows

Step 1: Set $w_k = x_k + \alpha(x_k - x_{k-1}) + \beta(x_{k-1} - x_{k-2})$ and compute

$$y_k = P_C\left(w_k - \lambda_k A w_k\right),$$

where $\lambda_k = min(\lambda_k^{(1)}, \lambda_k^{(2)})$

$$\lambda_k^{(1)} = \begin{cases} \min\left\{\frac{\mu(\|w_k - y_k\|)}{||Aw_k - A(y_k||}, \lambda_{k-1}\right\}, & \text{if } Aw_k - Ay_k \neq 0, \\ \lambda_{k-1}, & \text{otherwise .} \end{cases} \tag{3.1}$$

$\lambda_k^{(2)} = \gamma l^{m_k}$ *and* m_k *is the smallest nonnegative integer m such that*

$$\lambda_k^{(2)} \|Aw_k - Ay_k\| \leq \mu \|w_k - y_k\|. \tag{3.2}$$

If $y_k = w_k$ *then stop:* y_k *is a solution of the problem (VIP). Otherwise,*

Step 2: Compute

$$x_{k+1} = y_k - \lambda_k(Ay_k - Aw_k),$$

Set $k := k + 1$ *and go to* ***Step 1****.*

4. Convergence Analysis

Lemma 4.1. Under the Assumption (i)–(iv), the sequence λ_k generated Algorithm 3 satisfying

$$\min\{\frac{l\mu}{L}, \lambda_0\} \leq \lambda_k \leq \lambda_0. \tag{4.1}$$

Proof. Firstly, it is obvious that $\lambda_k^{(1)}$ is a monotonically decreasing sequence. Since A is a Lipschitz continuous mapping with constant $L > 0$, in the case of $Aw_k - Ay_k \neq 0$, we have

$$\frac{\mu \|w_k - y_k\|}{||Aw_k - Ay_k||} \geq \frac{\mu}{L} \frac{\|w_k - y_k\|}{\|w_k - y_k\|} = \frac{\mu}{L}$$

□

which implies that $0 < \min(\frac{l\mu}{L}, \lambda_0) < \min(\frac{\mu}{L}, \lambda_0) \leq \lambda_k^{(1)} \leq \lambda_0$. On the other hand, from the definition of $\lambda_k^{(2)}$ we have

$$\frac{\lambda_k^{(2)}}{l} \left\|Aw_k - Ay_k\right\| \geq \mu \left\|w_k - y_k\right\|$$

combining this with A is Lipschitz continuous on $\mathcal{H}$, we obtain

$$\frac{\lambda_k^{(2)}}{l} L \|w_k - y_k\| \geq \mu \|w_k - y_k\|$$

so $\lambda_k^{(2)} \geq \frac{l\mu}{L}$, therefore $\min\{\frac{l\mu}{L}, \lambda_0\} \leq \lambda_k \leq \lambda_0$.

Lemma 4.2. Let $\{x_k\}$ be a sequence generated by Algorithm 3. Then, $\{x_k\}$ is bounded.

Proof. Let $q \in S_D$. First of all, we estimate $||x_{k+1} - q||^2$. It can be obtained from the definition of x_{k+1} that:

$$\begin{aligned}
||x_{k+1} - q||^2 =& ||y_k - \lambda_k(Ay_k - Aw_k) - q||^2 \\
=& ||y_k - q||^2 + \lambda_k^2||Ay_k - Aw_k||^2 - 2\lambda_k\langle Ay_k - Aw_k, y_k - q\rangle \\
=& ||w_k - q||^2 + ||w_k - y_k||^2 + 2\langle y_k - w_k, w_k - q\rangle + \lambda_k^2||Ay_k - Aw_k||^2 \\
& - 2\lambda_k\langle Ay_k - Aw_k, y_k - q\rangle \\
=& ||w_k - q||^2 + ||w_k - y_k||^2 - 2\langle y_k - w_k, y_k - w_k\rangle + 2\langle y_k - w_k, y_k - q\rangle \\
& + \lambda_k^2||Ay_k - Aw_k||^2 - 2\lambda_k\langle Ay_k - Aw_k, y_k - q\rangle \\
=& ||w_k - q||^2 - ||w_k - y_k||^2 + 2\langle y_k - w_k, y_k - q\rangle + \lambda_k^2||Ay_k - Aw_k||^2 \\
& - 2\lambda_k\langle Ay_k - Aw_k, y_k - q\rangle. \qquad (4.2)
\end{aligned}$$

From the definition of y_k, and the fact that $S_D \subset C$, it can be obtained by Lemma 2.3 that:

$$\langle w_k - \lambda_k Aw_k - y_k, y_k - q\rangle \geq 0,$$

that is

$$\langle y_k - w_k, y_k - q\rangle \leq -\lambda_k\langle Aw_k, y_k - q\rangle. \qquad (4.3)$$

Appling (4.3) to (4.2), we will have

$$\begin{aligned}
||x_{k+1} - q||^2 \leq& ||w_k - q||^2 - ||w_k - y_k||^2 - 2\lambda_k\langle Aw_k, y_k - q\rangle \\
& + \lambda_k^2||Ay_k - Aw_k||^2 - 2\lambda_k\langle Ay_k - Aw_k, y_k - q\rangle \\
=& ||w_k - q||^2 - ||w_k - y_k||^2 + \lambda_k^2||Ay_k - Aw_k||^2 \\
& - 2\lambda_k\langle Ay_k, y_k - q\rangle. \qquad (4.4)
\end{aligned}$$

Since $q \in S_D, \lambda_k > 0$ and $y_k \in C$, we have that

$$\lambda_k\langle Ay_k, y_k - q\rangle \geq 0. \qquad (4.5)$$

Therefore, from (4.4) we can have

$$||x_{k+1} - q||^2 \leq ||w_k - q||^2 - ||w_k - y_k||^2 + \lambda_k^2||Ay_k - Aw_k||^2. \qquad (4.6)$$

From the definition of λ_k, if $Aw_k - Ay_k \neq 0$, then we have

$$\lambda_k^{(1)} = \min\{\frac{\mu||w_k - y_k||}{||Aw_k - Ay_k||}, \lambda_{k-1}\}$$

which means:

$$\lambda_k^{(1)} \leq \frac{\mu||w_k - y_k||}{||Aw_k - Ay_k||},$$

$$\lambda_k^{(1)}||Aw_k - Ay_k|| \leq \mu||w_k - y_k||.$$

Also, we have $\lambda_k^{(2)}\,\|Aw_k - Ay_k\| \leq \mu\,\|w_k - y_k\|$. Therefore,

$$\lambda_k\,\|Aw_k - Ay_k\| \leq \mu\,\|w_k - y_k\|\,. \qquad (4.7)$$

Substituting (4.7) into (4.6), we have

$$\begin{aligned}
||x_{k+1} - q||^2 \leq& ||w_k - q||^2 - ||w_k - y_k||^2 + \mu^2||w_k - y_k||^2 \\
=& ||w_k - q||^2 - (1 - \mu^2)||w_k - y_k||^2. \qquad (4.8)
\end{aligned}$$

Observe that

$$\begin{aligned}||x_{k+1}-y_k|| =&||y_k-\lambda_k(Ay_k-Aw_k)-y_k||\\ =&\lambda_k||Ay_k-Aw_k||\\ \leq&\mu||w_k-y_k||,\end{aligned}$$

and

$$\begin{aligned}||x_{k+1}-w_k|| \leq&||x_{k+1}-y_k||+||w_k-y_k||\\ \leq&\mu||w_k-y_k||+||w_k-y_k||\\ =&(1+\mu)||w_k-y_k||.\end{aligned} \tag{4.9}$$

Hence, using (4.9) into (4.8), we obtain that

$$-||w_k-y_k||^2 \leq -\frac{1}{(1+\mu)^2}||x_{k+1}-w_k||^2. \tag{4.10}$$

Applying (4.10) into (4.8), we have

$$||x_{k+1}-q||^2 \leq ||w_k-q||^2-(\frac{1-\mu}{1+\mu})||x_{k+1}-w_k||^2. \tag{4.11}$$

Also,

$$\begin{aligned}w_k-q =&x_k+\alpha(x_k-x_{k-1})+\beta(x_{k-1}-x_{k-2})-q\\ =&(1+\alpha)(x_k-q)-(\alpha-\beta)(x_{k-1}-q)-\beta(x_{k-2}-q).\end{aligned}$$

Using Lemma 2.2, we get

$$\begin{aligned}||w_k-q||^2 =&||(1+\alpha)(x_k-q)-(\alpha-\beta)(x_{k-1}-q)-\beta(x_{k-2}-q)||^2\\ =&(1+\alpha)||x_k-q||^2-(\alpha-\beta)||x_{k-1}-q||^2-\beta||x_{k-2}-q||^2\\ &+(1+\alpha)(\alpha-\beta)||x_k-x_{k-1}||^2+\beta(1+\alpha)||x_k-x_{k-2}||\\ &-\beta(\alpha-\beta)||x_{k-1}-x_{k-2}||^2.\end{aligned} \tag{4.12}$$

Furthermore, it can be obtained that

$$\begin{aligned}||x_{k+1}-w_k||^2 =&\|x_{k+1}-[x_k+\alpha(x_k-x_{k-1})+\beta(x_{k-1}-x_{k-2})]\|^2\\ =&\|x_{k+1}-x_k-\alpha(x_k-x_{k-1})-\beta(x_{k-1}-x_{k-2})\|^2\\ =&\|x_{k+1}-x_k\|^2-2\alpha\langle x_{k+1}-x_k,x_k-x_{k-1}\rangle-2\beta\langle x_{k+1}-x_k,x_{k-1}-x_{k-2}\rangle\\ &+\alpha^2||x_k-x_{k-1}||^2+2\alpha\beta\langle x_k-x_{k-1},x_{k-1}-x_{k-2}\rangle+\beta^2||x_{k-1}-x_{k-2}||\\ \geq&\|x_{k+1}-x_k\|^2-\alpha\|x_{k+1}-x_k\|^2-\alpha\|x_k-x_{k-1}\|^2+\alpha^2\|x_k-x_{k-1}\|\\ &-|\beta|\alpha\|x_k-x_{k-1}\|^2-|\beta|\alpha\|x_{k-1}-x_{k-2}\|^2+\beta^2\|x_{k-1}-x_{k-2}\|^2\\ &-|\beta|\|x_{k+1}-x_k\|^2-|\beta|\|x_{k-1}-x_{k-2}\|^2\\ =&(1-|\beta|-\alpha)\|x_{k+1}-x_k\|^2+(\alpha^2-\alpha-|\beta|\alpha)\|x_k-x_{k-1}\|^2\\ &+(\beta^2-|\beta|-|\beta|\alpha)\|x_{k-1}-x_{k-2}\|^2.\end{aligned} \tag{4.13}$$

Applying (4.12) and (4.13) with (4.11) and note that $\beta < 0$ we have

$$\begin{aligned}
||x_{k+1} - q||^2 \leq & (1+\alpha)\|x_k - q\|^2 - (\alpha - \beta)\|x_{k-1} - q\|^2 - \beta\|x_{k-2} - q\|^2 \\
& + (1+\alpha)(\alpha - \beta)\|x_k - x_{k-1}\|^2 + \beta(1+\alpha)\|x_k - x_{k-2}\|^2 \\
& - \beta(\alpha - \beta)\|x_{k-1} - x_{k-2}\|^2 - \Big(\frac{1-\mu}{1+\mu}\Big)(1 - |\beta| - \alpha)\|x_{k+1} - x_k\|^2 \\
& - \Big(\frac{1-\mu}{1+\mu}\Big)(\alpha^2 - \alpha - |\beta|\alpha)\|x_k - x_{k-1}\|^2 \\
& - \Big(\frac{1-\mu}{1+\mu}\Big)(\beta^2 - |\beta| - |\beta|\alpha)\|x_{k-1} - x_{k-2}\|^2 \\
\leq & (1+\alpha)\|x_k - q\|^2 - (\alpha - \beta)\|x_{k-1} - q\|^2 - \beta\|x_{k-2} - q\|^2 + \Big[(1+\alpha)(\alpha - \beta) \\
& - \Big(\frac{1-\mu}{1+\mu}\Big)(\alpha^2 - \alpha + \beta\alpha)\Big]\|x_k - x_{k-1}\|^2 \\
& - [\beta(\alpha - \beta) + \Big(\frac{1-\mu}{1+\mu}\Big)(\beta^2 + \beta + \beta\alpha)]\|x_{k-1} - x_{k-2}\|^2 \\
& - \Big(\frac{1-\mu}{1+\mu}\Big)(1 + \beta - \alpha)\|x_{k+1} - x_k\|^2. \qquad (4.14)
\end{aligned}$$

By rearranging the inequality above we obtain that:

$$\begin{aligned}
& \|x_{k+1} - q\|^2 - \alpha\|x_k - q\|^2 - \beta\|x_{k-1} - q\|^2 + \Big(\frac{1-\mu}{1+\mu}\Big)(1 + \beta - \alpha)\|x_{k+1} - x_k\|^2 \\
\leq & \|x_k - q\|^2 - \alpha\|x_{k-1} - q\|^2 - \beta\|x_{k-2} - q\|^2 + \Big(\frac{1-\mu}{1+\mu}\Big)(1 + \beta - \alpha)\|x_k - x_{k-1}\|^2 \\
& + \big[(1+\alpha)(\alpha - \beta) - \big(\frac{1-\mu}{1+\mu}\big)(\alpha^2 - 2\alpha + \beta\alpha + \beta + 1)\big]\|x_k - x_{k-1}\|^2 \\
& - [\beta(\alpha - \beta) + \left(\frac{1-\mu}{1+\mu}\right)(\beta^2 + \beta + \beta\alpha)]\|x_{k-1} - x_{k-2}\|^2. \qquad (4.15)
\end{aligned}$$

Now, let

$$\Gamma_k = \|x_k - q\|^2 - \alpha\|x_{k-1} - q\|^2 - \beta\|x_{k-2} - q\|^2 + \left(\frac{1-\mu}{1+\mu}\right)(1 + \beta - \alpha)\|x_k - x_{k-1}\|^2.$$

Then we can rewrite (4.15) as

$$\begin{aligned}
\Gamma_{k+1} \leq & \Gamma_k + [(1+\alpha)(\alpha - \beta) - \left(\frac{1-\mu}{1+\mu}\right)(\alpha^2 - 2\alpha + \beta\alpha + \beta + 1)]\|x_k - x_{k-1}\|^2 \\
& - [\beta(\alpha - \beta) + \left(\frac{1-\mu}{1+\mu}\right)(\beta^2 + \beta + \beta\alpha)]\|x_{k-1} - x_{k-2}\|^2. \qquad (4.16)
\end{aligned}$$

Claim: $\Gamma_k \geq 0, \forall k > 1$. Since

$$\begin{aligned}
\Gamma_k = & \|x_k - q\|^2 - \alpha\|x_{k-1} - q\|^2 - \beta\|x_{k-2} - q\|^2 + \left(\frac{1-\mu}{1+\mu}\right)(1 + \beta - \alpha)\|x_k - x_{k-1}\|^2 \\
\geq & \|x_k - q\|^2 - 2\alpha\|x_k - x_{k-1}\|^2 - 2\alpha\|x_k - q\|^2 - \beta\|x_{k-2} - q\|^2 \\
& + \left(\frac{1-\mu}{1+\mu}\right)(1 + \beta - \alpha)\|x_k - x_{k-1}\|^2 \\
= & (1 - 2\alpha)\|x_k - q\|^2 + \left[\left(\frac{1-\mu}{1+\mu}\right)(1 + \beta - \alpha) - 2\alpha\right]\|x_k - x_{k-1}\|^2 \\
& - \beta||x_{k-2} - q||^2. \qquad (4.17)
\end{aligned}$$

From Assumption 2, we have that $\alpha < \frac{1}{2}, \beta \leq 0$ and $0 < \alpha < \frac{1-\mu}{3+\mu}$. Tt can be conclude that $\Gamma_k \geq 0$, $\forall k > 1$. Let

$$\begin{aligned} k_1 :=& -\left[(1+\alpha)(\alpha-\beta) - \left(\frac{1-\mu}{1+\mu}\right)(\alpha^2 - 2\alpha + \beta\alpha + \beta + 1)\right] \\ k_2 :=& -\left[(1+\alpha)(\alpha-\beta) - \left(\frac{1-\mu}{1+\mu}\right)(\alpha^2 - 2\alpha + \beta\alpha + \beta + 1) - \beta(\alpha-\beta)\right. \\ & \left. - \left(\frac{1-\mu}{1+\mu}\right)(\beta^2 + \beta + \beta\alpha)\right]. \end{aligned} \tag{4.18}$$

Thus, we deduce from (4.16) that

$$\Gamma_{k+1} - \Gamma_k \leq k_1(\|x_{k-1} - x_{k-2}\|^2 - \|x_k - x_{k-1}\|^2) - k_2\|x_{k-1} - x_{k-2}\|^2. \tag{4.19}$$

From Assumption 2(b),

$$\max\left\{2\alpha(\frac{1-\mu}{3+\mu}) - (1-\alpha), \frac{1}{2}[\alpha(1+\mu) - (\frac{(1-\mu)(1-\alpha)^2}{1+\alpha})]\right\} < \beta \leq 0,$$

which implies that

$$\frac{1}{2}\left[\alpha(1+\mu) - \frac{(1-\mu)(1-\alpha)^2}{1+\alpha}\right] < \beta, \tag{4.20}$$

which can conclude that $k_1 > 0$.
By Assumption 2(c) we can have $k_2 > 0$, thus, (4.19) can be rewritten as

$$\Gamma_{k+1} + k_1\|x_k - x_{k-1}\|^2 \leq \Gamma_k + k_1\|x_{k-1} - x_{k-2}\|^2 - k_2\|x_{k-1} - x_{k-2}\|^2. \tag{4.21}$$

Letting $\Gamma_k' = \Gamma_k + k_1\|x_{k-1} - x_{k-2}\|^2$. Then, $\Gamma_k' \geq 0, \forall k \geq 1$. Therefore, we deduce from (4.21) that

$$\Gamma_{k+1}' \leq \Gamma_k', \tag{4.22}$$

which implies that the sequence Γ_{k+1}' is decreasing and bounded from below and thus $\lim_{n\to\infty} \Gamma_k'$ exists. Hence, by rearranging (4.21) and letting k approach infinity, we have

$$\lim_{k\to\infty} k_2||x_{k-1} - x_{k-2}||^2 = 0 \implies \lim_{k\to\infty} ||x_{k-1} - x_{k-2}|| = 0. \tag{4.23}$$

It is easy to see that

$$\begin{aligned} \|x_{k+1} - w_k\| =& \|x_{k+1} - x_k - \alpha(x_k - x_{k-1}) - \beta(x_{k-1} - x_{k-2})\| \\ \leq& \|x_{k+1} - x_k\| + \alpha\|x_k - x_{k-1}\| + \beta\|x_{k-1} - x_{k-2}\|. \end{aligned} \tag{4.24}$$

Consequently, by (4.23) we have

$$\lim_{k\to\infty} \|x_{k+1} - w_k\| = 0. \tag{4.25}$$

Furthermore,

$$\begin{aligned} \|x_k - w_k\| =& \|x_k - x_k - \alpha(x_k - x_{k-1}) - \beta(x_{k-1} - x_{k-2})\| \\ \leq& \alpha\|x_k - x_{k-1}\| + \beta\|x_{k-1} - x_{k-2}\| \to 0, k \to \infty. \end{aligned} \tag{4.26}$$

Observe that

$$\begin{aligned}\|x_{k+1}-w_k\| &= \|y_k-\lambda_k(Ay_k-Aw_k)-w_k\| \\ &\geq \|y_k-w_k\|-\lambda_k\|Ay_k-Aw_k\| \\ &\geq \|y_k-w_k\|-\mu\|y_k-w_k\| \\ &= (1-\mu)\|y_k-w_k\|. \end{aligned} \tag{4.27}$$

Using (4.25)we deduce from (4.27) that

$$\lim_{k\to\infty}\|w_k-y_k\|=0. \tag{4.28}$$

Also,

$$\|x_k-y_k\|\leq\|x_k-w_k\|+\|w_k-y_k\|\to 0, k\to\infty. \tag{4.29}$$

Due to the existence of the limit $\Gamma_k^{'}$ and (4.23), we can obtain that the limit of Γ_k also exists and therefore, the sequence $\{\Gamma_k\}$ is bounded.
Since $\lim_{k\to\infty}||x_{k+1}-x_k||=0$, from the definition of Γ_k we can get that

$$\lim_{k\to\infty}(\|x_k-q\|^2-\alpha\|x_{k-1}-q\|^2-\beta\|x_{k-2}-q\|^2) \tag{4.30}$$

exists. Due to the boundedness of $\{\Gamma_k\}$, we have from (4.17) that $\{x_k\}$ is bounded and $\{y_k\}$ and $\{w_k\}$ are bounded. □

Lemma 4.3. Let $\{x_k\}$ be the sequence generated by Algorithm 3, satisfying Assumptions 1 (i)-(iv) and Assumptions 2 (a)-(c), and assuming that $\lim_{k\to\infty}(w_k-y_k)=0$. If $v^*$is one of the weakly clustered points of $\{y_k\}$, then we have at least one of the following: $v^*\in S_D$ or $Av^*=0$.

Proof. By Lemma 4.2 above, it can be concluded that $\{y_k\}$ is bounded. Therefore, let v^* be a weak cluster point of $\{y_k\}$. Hence, we denote $\{y_{k_j}\}$ as a subsequence of $\{y_k\}$ such that $y_{k_j}\rightharpoonup v^*\in C$. Now we discuss in two cases.

Case 1. First, we assume that $\lim_{j\to\infty}||Ay_{k_j}||=0$. Consequently, $\lim_{j\to\infty}||Ay_{k_j}||=\liminf_{j\to\infty}||Ay_{k_j}||=0$. Through the assumption above that $y_{k_j}\rightharpoonup v^*\in C$ and A satisfies Assumption 1 (iii), that is

$$0\leq||Av^*||\leq\liminf_{j\to\infty}||Ay_{k_j}||=0. \tag{4.31}$$

Which just implies that $Av^*=0$.

Now we consider the another situation.

Case 2. If $\limsup_{j\to\infty}\|Ay_{k_j}\|>0$. Without loss of generality, we take $\lim_{j\to\infty}\|Ay_{k_j}\|=M_1>0$. It then follows that there exists a $\mathbf{K}\in\mathbb{N}$ such that $\|Ay_{k_j}\|>\frac{M_1}{2}$, for all $j\geq\mathbf{K}$. Since $y_{k_j}=P_C(w_{k_j}-\lambda_{k_j}Aw_{k_j})$, we have

$$\langle w_{k_j}-\lambda_{k_j}Aw_{k_j}-y_{k_j},x-y_{k_j}\rangle\leq 0.$$

$$\text{Hence, } \frac{1}{\lambda_{k_j}}\langle w_{k_j}-y_{k_j},x-y_{k_j}\rangle+\langle Aw_{k_j},y_{k_j}-w_{k_j}\rangle\leq\langle Aw_{k_j},x-w_{k_j}\rangle. \tag{4.32}$$

For the weak convergence of $\{w_{k_j}\}$, since $\{w_{k_j}\}$ is bounded. Then, by the Lipschitz continuity of A, $\{Aw_{k_j}\}$ is bounded. By (4.28) we can get that

$$\|w_{k_j}-y_{k_j}\|\to 0, j\to\infty.$$

Therefore, $\{y_{k_j}\}$ is also bounded and using Lemma 4.1 we get that $\lambda_{k_j} \geq \min\{\frac{l\mu}{L}, \lambda_0\}$. Passing (4.32) to the limit as $j \to \infty$, that is

$$\liminf_{j\to\infty} \langle Aw_{k_j}, x - w_{k_j}\rangle \geq 0, \forall x \in C. \tag{4.33}$$

Observe that

$$\begin{aligned}\langle Ay_{k_j}, x - y_{k_j}\rangle =& \langle Ay_{k_j} - Aw_{k_j}, x - w_{k_j}\rangle + \langle Aw_{k_j}, x - w_{k_j}\rangle \\ &+ \langle Ay_{k_j}, w_{k_j} - y_{k_j}\rangle.\end{aligned} \tag{4.34}$$

Thus, we can get $\lim_{j\to\infty} ||Aw_{k_j} - Ay_{k_j}|| = 0$, for the $\lim_{j\to\infty} ||w_{k_j} - y_{k_j}|| = 0$ and the L -Lipschitz continuity on H of A. Together with (4.33) and (4.34) which implies that

$$\lim_{j\to\infty} \langle Ay_{k_j}, x - y_{k_j}\rangle \geq 0. \tag{4.35}$$

If we suppose that $\limsup_{j\to\infty} \langle Ay_{k_j}, x - y_{k_j}\rangle > 0$, then there exists a subsequence denoted by $\{y_{k_{j_n}}\}$ such that $\lim_{n\to\infty} \langle Ay_{k_{j_n}}, x - y_{k_{j_n}}\rangle > 0$, that is there exists $n_0 \in \mathbb{N}$ such that

$$\langle Ay_{k_{j_n}}, x - y_{k_j}\rangle \geq 0, \forall n > n_0. \tag{4.36}$$

By quasimonotonicity of A, we get $\langle Ax, x - y_{k_{j_n}}\rangle \geq 0$, as $n \to \infty$, we conclude that $v^* \in S_D$. On the other hand we suppose that if $\limsup_{j\to\infty} \langle Ay_{k_j}, x - y_{k_j}\rangle = 0$. Then, (4.33) implies that

$$\lim_{j\to\infty} \langle Ay_{k_j}, x - y_{k_j}\rangle = 0. \tag{4.37}$$

Let $\epsilon_j := |\langle Ay_{k_j}, x - y_{k_j}\rangle| + \frac{1}{j+1}$. Thus, we obtain that

$$\langle Ay_{k_j}, x - y_{k_j}\rangle + \epsilon_j > 0, \forall j \geq 1. \tag{4.38}$$

Furthermore, for each $j \geq \mathbf{K}$ we can get $Ay_{k_j} \neq 0$. Defining that

$$r_{k_j} = \frac{Ay_{k_j}}{||Ay_{k_j}||^2}, \forall\, j \geq \mathbf{K},$$

then $\langle Ay_{k_j}, r_{k_j}\rangle = 1$ for each $j \geq \mathbf{K}$. Thus, we can conclude from (4.38) that,

$$\langle Ay_{k_j}, x + \epsilon_j r_{k_j} - y_{k_j}\rangle > 0,\ j \geq \mathbf{K}.$$

Since A is quasimonotone on H, we get

$$\langle A(x + \epsilon_j r_{k_j}), x + \epsilon_j r_{k_j} - y_{k_j}\rangle \geq 0. \tag{4.39}$$

Thus,

$$\begin{aligned}\langle Ax, x + \epsilon_j r_{k_j} - y_{k_j}\rangle =& \langle Ax - A(x + \epsilon_j r_{k_j}), x + \epsilon_j r_{k_j} - y_{k_j}\rangle \\ &+ \langle A(x + \epsilon_j r_{k_j}), x + \epsilon_j r_{k_j} - y_{k_j}\rangle\rangle \\ \geq& \langle Ax - A(x + \epsilon_j r_{k_j}), x + \epsilon_j r_{k_j} - y_{k_j}\rangle \\ \geq& - \|Ax - A(x + \epsilon_k r_{k_j})\|\|x + \epsilon_k r_{k_j} - y_{k_j}\| \\ \geq& - \epsilon_j L\|r_{k_j}\|\|x + \epsilon_j r_{k_j} - y_{k_j}\| \\ =& - \epsilon_j L\frac{1}{\|Ay_{k_j}\|}\|x + \epsilon_j r_{k_j} - y_{k_j}\| \\ \geq& - \epsilon_j L\frac{2}{M_1}\|x + \epsilon_j r_{k_j} - y_{k_j}\|.\end{aligned} \tag{4.40}$$

Observe that, tending $j \to \infty$, for $\{x + \epsilon_j r_{k_j} - y_{k_j}\}$ is bounded and $\lim_{k\to\infty} \epsilon_j = 0$, we can conclude that $\langle Ax, x - v^*\rangle \geq 0,\ \forall x \in C$. This implies that $v^* \in S_D$.

□

Theorem 4.1. Suppose $\{x_k\}$ is a sequence generated by Algorithm 3. Then under Assumptions 1 and Assumptions 2 and $Ax \neq 0, \forall x \in C$. Then $\{x_k\}$ converges weakly to an element of $S_D \subset S$.

Proof. Suppose $w_w(x_k)$ is a set of weak cluster points of $\{x_k\}$. Then we show that $w_w(x_k)$ is a nonempty subset of S_D. Let $v^* \in w_w(x_k)$. Then, there exists a subsequence $\{x_{k_j}\} \subset \{x_k\}$ such that $x_{k_j} \to v^*$, as $j \to \infty$. Since C is weakly closed, we have that $v^* \in C$. Furthermore, we can conclude that $Av^* \neq 0$ since $Ax \neq 0$, $\forall x \in C$. By Lemma 4.3, we have $v^* \in S_D$. Hence $w_w(x_k) \subset S_D$. By Lemma 4.2, $\lim_{k\to\infty} \Gamma_k$ exists and $\lim_{k\to\infty} ||x_{k+1} - x_k|| = 0$, we have

$$\lim_{k\to\infty} [||x_k - q||^2 - \alpha||x_{k-1} - q||^2 - \beta||x_{k-1} - q||^2] \tag{4.41}$$

exists, $\forall q \in S_D$.

Next we show that $x_k \rightharpoonup x^* \in S_D$. Let $\{x_{k_j}\}$ and $\{x_{k_n}\}$ be two subsequences of $\{x_k\}$ such that $x_{k_j} \rightharpoonup v^*, j \to \infty$ and $x_{k_n} \rightharpoonup x^*, n \to \infty$.

Then we show that $x^* = v^*$, using (2.4) observe that

$$2\langle x_k, x^* - v^*\rangle = ||x_k - v^*||^2 - ||x_k - x^*||^2 - ||v^*||^2 + ||x^*||^2; \tag{4.42}$$

$$2\langle x_{k-1}, x^* - v^*\rangle = ||x_{k-1} - v^*||^2 - ||x_{k-1} - x^*||^2 - ||v^*||^2 + ||x^*||^2; \tag{4.43}$$

and

$$2\langle x_{k-2}, x^* - v^*\rangle = ||x_{k-2} - v^*||^2 - ||x_{k-2} - x^*||^2 - ||v^*||^2 + ||x^*||^2. \tag{4.44}$$

Therefore,

$$2\langle -\alpha x_{k-1}, x^* - v^*\rangle = -\alpha||x_{k-1} - v^*||^2 + \alpha||x_{k-1} - x^*||^2 + \alpha||v^*||^2 - \alpha||x^*||^2; \tag{4.45}$$

and

$$2\langle -\beta x_{k-1}, x^* - v^*\rangle = -\beta||x_{k-1} - v^*||^2 + \beta||x_{k-1} - x^*||^2 + \beta||v^*||^2 - \beta||x^*||^2; \tag{4.46}$$

Addition of (4.42), (4.45) and (4.46) gives

$$\begin{aligned} 2\langle x_k - \alpha x_{k-1} - \beta x_{k-2}, x^* - v^*\rangle = &\big(\|x_k - v^*\|^2 - \alpha\|x_{k-1} - v^*\|^2 - \beta\|x_{k-2} - v^*\|^2\big) \\ &- \big(\|x_k - x^*\|^2 - \alpha\|x_{k-1} - x^*\|^2 - \beta\|x_{k-2} - x^*\|^2\big) \\ &+ (1 - \alpha - \beta)(\|x^*\| - \|v^*\|^2). \end{aligned} \tag{4.47}$$

From (4.30) we know that

$$\lim_{k\to\infty} \big(\|x_k - x^*\|^2 - \alpha\|x_{k-1} - x^*\|^2 - \beta\|x_{k-2} - x^*\|^2\big) \tag{4.48}$$

exists, and

$$\lim_{k\to\infty} \big(\|x_k - v^*\|^2 - \alpha\|x_{k-1} - x^*\|^2 - \beta\|x_{k-2} - x^*\|^2\big) \tag{4.49}$$

also exists, which implies with (4.47) that $\lim_{k\to\infty} \langle x_k - \alpha x_{k-1} - \beta x_{k-2}, x^* - v^*\rangle$ exists.

Consequently,

$$\begin{aligned} \langle v^* - \alpha v^* - \beta v^*, x^* - v^*\rangle &= \lim_{j\to\infty} \langle x_{k_j} - \alpha x_{k_{j-1}} - \beta x_{n_{j-2}}, x^* - v^*\rangle \\ &= \lim_{k\to\infty} \langle x_k - \alpha x_{k-1} - \beta x_{k-2}, x^* - v^*\rangle \\ &= \lim_{n\to\infty} \langle x_{k_n} - \alpha x_{k_{n-1}} - \beta x_{k_{n-2}}, x^* - v^*\rangle \\ &= \langle x^* - \alpha x^* - \beta x^*, x^* - v^*\rangle. \end{aligned} \tag{4.50}$$

Thus,

$$(1 - \alpha - \beta)||x^* - v^*||^2 = 0. \tag{4.51}$$

Since $\beta \leq 0 < 1-\alpha$, $1-\alpha-\beta \neq 0$, we can conclude that $x^* = v^*$. Hence, we deduce that $\{x_k\}$ converges weakly to a point in S_D. This completes the proof. □

5. Conclusions

This paper proposed a novel two-step inertial Tseng extragradient method for solving quasimonotone variational inequalities in real Hilbert spaces. The algorithm incorporates a dual step-size strategy, adaptively selecting between a self-adaptive rule and an Armijo-like rule at each iteration.The proposed method effectively addresses limitations of existing extragradient methods, particularly the need for Lipschitz constants and the computational burden of multiple projections.

Statements and Declarations

All authors have contributed to the preparation of this manuscript and have approved its submission forpublication. The order of authors listed in the manuscript has been agreed upon by all authors.

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