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CONVERGENCE RATE OF THE AVERAGED CQ ALGORITHMS UNDER HÖLDERIAN TYPE BOUNDED LINEAR REGULARITY PROPERTY

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ABSTRACT. We study the issue of the strong convergence together with estimates of convergence order of an averaged CQ algorithm for solving the split feasibility problem in Hilbert space. For this purpose, a Hölderian type bounded linear regularity property is introduced. When the involved parameters and stepsizes satisfy certain mild conditions, the strong convergence together with estimates of convergence order of the averaged CQ algorithm is established under the Hölderian type bounded linear regularity property. For the case when the involved parameters are all equal to constant 1, the averaged CQ algorithm is reduced to the well-known CQ algorithm. As applications, we obtain the strong convergence together with estimates of convergence order of the CQ algorithm, which extends the corresponding ones in (Wang, et al. Inverse Problem, 2017, 33: 055017). Finally, numerical experiments are presented to illustrate the effectiveness of the algorithm. Compared to other known algorithms, our algorithm performs better.

Keywords. Split feasibility problems, Averaged CQ algorithm, Strong convergence, Estimate of convergence rate, Hilbert space.

1. INTRODUCTION

Let H_1 and H_2 be real Hilbert spaces. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (SFP for short) is formulated as follows: finding a point $x \in H_1$ such that

$$x \in C \quad \text{and} \quad Ax \in Q. \tag{1.1}$$

We use S to denote the solution set of the SFP (1.1). Throughout the whole paper, we always assume that S is nonempty, that is,

$$S := C \cap A^{-1}Q \neq \emptyset.$$

The SFP (1.1) was introduced by Censor and Elfving in [9] for solving the phase retrieval problem, which provides a unified framework for the study of many inverse problems and has important applications in various areas, such as signal processing, image reconstruction and intensity-modulated radiation therapy[6, 12, 20, 11, 10]. Many algorithms have been developed to solve the SFP (1.1). One of the most famous and practical algorithms is the CQ algorithm which was given by Byrne[6, 5], and has the following iterative form:

$$x_{n+1} = P_C\left(x_n - \beta A^*\left(I - P_Q\right)Ax_n\right), \tag{1.2}$$

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where $\beta \geq 0$ is the stepsizes, I is the identity, A^* is the adjoint of A, while P_C and P_Q are the metric projections onto sets C and Q, respectively. The CQ algorithm with different types of stepsizes and various variants of the CQ algorithm have been extensively explored; see [7, 28, 30, 21, 24, 26, 27] and references therein. In particular, in Hilbert space, the weak convergence of the CQ algorithm with constant stepsizes was established in [27] by virtue of the theory of fixed points. López et al. [21] introduced the CQ algorithm with dynamic stepsizes in Hilbert spaces and established the weak convergence. The CQ algorithm with dynamic stepsizes has the advantage that it does not require any prior knowledge about the norm of operator (matrix) A.

In general, the CQ algorithms with dynamic or constant stepsizes might not converge strongly in Hilbert spaces (see [27, Example 3.7] or [21, Proposition 7]). Wang et al. [25] established the linear convergence result for the CQ algorithm with the constant or dynamic stepsizes under the bounded linear regularity property in Hilbert space. On the other hand, the strong convergence together with an estimate of convergence rate of the relaxed CQ algorithm was established in [29] under Hölderian error bound property.

Note that Xu [27] also proposed an averaged CQ algorithm in Hilbert space by combining the CQ algorithm with Mann's algorithm (see [27, (3.14)]), which is formulated as follows:

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n P_C\left(x_n - \gamma A^*\left(I - P_Q\right) A x_n\right),$$

where $0 < \gamma < \frac{2}{\|A\|^2}$ and $0 \leq \alpha_n \leq \frac{4}{2+\gamma\|A\|^2}$. The weak convergence of the averaged CQ algorithm was established in [27]. As pointed out in [27], the averaged CQ algorithm might not converge strongly (see[27, Example 3.7]). In this paper, we continue to study the averaged CQ algorithm with constant stepsize or dynamic stepsize, which is stated as follows. Throughout the whole paper, we always adopt the convention that $\frac{0}{0} = 0$.

Algorithm 1.1 Let $x_0 \in C$ be given. Having $x_0, x_1, \cdots, x_n$, choose a parameter $0 \leq \alpha_n \leq 1$ and a stepsizes $\beta_n > 0$, and determine x_{n+1} by

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n P_C\left(x_n - \beta_n A^*\left(I - P_Q\right) A x_n\right).$$

Clearly, if $\alpha_n = 1$ for each n, Algorithm 1.1 is reduced to the CQ algorithm (1.2). Consider three different kinds of stepsizes:

$$\sigma_1 \leq \beta_n \leq \frac{2}{\|A\|^2} - \sigma_1 \quad \text{with} \quad 0 < \sigma_1 < \frac{1}{\|A\|^2} - \sigma_1; \tag{1.3}$$

$$\beta_n = \frac{\rho_n \left\|\left(I - P_Q\right) A x_n\right\|^2}{\left\|A^*\left(I - P_Q\right) A x_n\right\|^2} \quad \text{with} \quad \sigma_2 < \rho_n < 2 - \sigma_2 \text{ and } 0 < \sigma_2 < 1; \tag{1.4}$$

$$\lim_{n\to\infty} \beta_n = 0. \tag{1.5}$$

Wang et al. in [25] established the strong convergence of the CQ algorithm (1.2) with stepsize $\{\beta_n\}$ satisfying (1.3) or (1.4), or (1.5) and $\sum_n \beta_n = \infty$. The weak convergence of the CQ algorithm (1.2) was studied in [27] (resp. [21]) with stepsize $\{\beta_n\}$ satisfying (1.3) (resp. (1.4)).

To the best of our knowledge, the study of the strong convergence of the averaged CQ algorithm with the stepsize $\{\beta_n\}$ satisfying (1.3) or (1.4) or (1.5), is very limited. Motivated by the works of [25] and [29], the main purpose of the present paper is to study the strong convergence together with estimates of convergence rate of Algorithm 1.1. Note that the regularity condition plays a key role in the convergence analysis of many algorithms [1, 4, 13, 19]. A bounded linear regularity property was introduced by Wang et al. [25] to establish the linear convergence of the CQ algorithm for the SFP(1.1). A natural extension of the bounded linear regularity property is the bounded linear regularity property with fractional exponent, that is, the Hölderian type bounded linear regularity property. The exponent is closely related to the estimates of the convergence order of some algorithms; see [3, 15, 14, 16, 17, 18, 22]. In order to explore the strong convergence together with estimates of convergence order of Algorithm

1.1, we introduce the bounded linear regularity property with exponent τ $(0 < \tau \leq 1)$. Under the bounded linear regularity property with exponent τ, the strong convergence together with estimates of convergence rate of Algorithm 1.1 is obtained with the stepsize $\{\beta_n\}$ satisfying (1.3) or (1.4) or (1.5), and $\sum_n \alpha_n\beta_n = \infty$; see Theorem 3.1. The stepsize of Algorithm 1.1 can also switch between (1.3) and (1.4) for different n. In this case, the strong convergence together with estimates of convergence rate of Algorithm 1.1 is also provided; see Theorem 3.2. For the case when $\alpha_n = 1$ for each n, Algorithm 1.1 is reduced to the CQ algorithm (1.2). As applications, the strong convergence together with estimates of convergence rate of the CQ algorithm is presented under the bounded linear regularity property with exponent τ (see Corollary 3.3 and Corollary 3.4), which extends the corresponding one in [25, Theorem 2.3]. Finally, numerical experiments are provided to illustrate the effectiveness of the algorithm. Compared to other known algorithms, our algorithm performs better.

The rest of the paper is organised as follows. Some notation and preliminary results are given in the next section. Section 3 investigates the strong convergence together with estimates of convergence rate of Algorithm 1.1 under the bounded linear regularity property with exponent τ. Some numerical experiments are provided in Section 4.

2. Preliminaries

In what follows, let $\mathbb{N}$ denote the set of all positive integers, and let $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and its associated norm $\|\cdot\|$. Given $x \in H$ and $r > 0$, we use $\mathbf{B}(x,r)$ and $\overline{\mathbf{B}(x,r)}$ to denote the open metric ball and the closed metric ball centered at x with radius r, respectively. Let $\Omega \subset H$. The distance function of Ω and the projection onto Ω are denoted by $\mathrm{d}_\Omega(\cdot)$ and $P_\Omega(\cdot)$, and are defined by

$$\mathrm{d}_\Omega(x) := \inf_{y\in\Omega} \|x-y\| \quad \text{and} \quad P_\Omega(x) := \{y \in \Omega :\ \mathrm{d}_\Omega(x) = \|x-y\|\} \qquad \text{for each } x \in H,$$

respectively. Let I be the identity operator on H.

The following lemma is about some useful properties of projection operators, where (i) is taken from [2, Proposition 4.2(i)], while (ii) follows from [2, Proposition 4.2(ii)] and [2, Corollary 4.10].

Lemma 2.1. *Let Ω be a nonempty closed convex subset of H. Then the following two assertions hold:*

(i) *The operator P_Ω is nonexpansive, i.e., $\|P_\Omega x - P_\Omega y\| \leq \|x-y\|$ for all $x, y \in H$.*
(ii) $\langle (I-P_\Omega)x - (I-P_\Omega)y, x-y\rangle \geq \|(I-P_\Omega)x - (I-P_\Omega)y\|^2$ *for all $x, y \in H$.*

The following lemma is taken from [23, Lemma 6], which will be useful in our convergence analysis of the averaged CQ algorithms.

Lemma 2.2. *Let $p > 0$, and let $\{\alpha_k\}$ and $\{\mu_k\}$ be nonnegative sequences satisfying:*

$$\mu_{k+1} \leq \mu_k(1-\alpha_k\mu_k^p), \quad \forall k \in \mathbb{N}^*.$$

Then,

$$u_{k+1} \leq \left(u_0^{-p} + p\sum_{i=0}^{k}\alpha_i\right)^{-\frac{1}{p}}, \quad \forall k \in \mathbb{N}^*.$$

Regularity conditions play a crucial role in the convergence analysis of many algorithms [1, 4, 13, 19]. To establish the linear convergence of the CQ algorithm for the SFP (1.1), Wang et al. [25] introduced the bounded regularity condition for the SFP (1.1). A natural extension of the bounded regularity condition is the fractional exponent bounded regularity condition, i.e., the Hölderian-type error bound. The exponent constant is related to the estimation of the convergence rate of some algorithms; see [3, 15, 14, 16, 17, 18, 22]. To study the strong convergence together with estimates of convergence rate of Algorithm 1.1, we introduce the following bounded regularity condition with exponent τ. Recall that S is the solution set of the SFP (1.1).

Definition 2.3. Let $0 < \tau \leq 1$. The SFP (1.1) is said to satisfy the bounded linear regularity property with exponent τ if for any $r > 0$ with $S \cap \overline{\mathbf{B}(0,r)} \neq \emptyset$, there exists $\gamma_r > 0$ such that

$$\gamma_r \mathrm{d}_S(x) \leq \mathrm{d}_Q^\tau(Ax), \quad \forall x \in C \cap \overline{\mathbf{B}(0,r)}. \tag{2.1}$$

In particular, if $\tau = 1$, the SFP (1.1) is said to satisfy the bounded linear regularity property.

3. Convergence of The Averaged CQ Algorithm

This section is devoted to studying the strong convergence together with estimates of convergence rate of the averaged CQ algorithm. Recall that $S = C \cap A^{-1}Q$. Then the following equivalence is trivial:

$$[z \in S] \iff [(I - P_Q)Az = 0], \quad \forall z \in C. \tag{3.1}$$

Theorem 3.1 below shows that a sequence generated by Algorithm 1.1 converges strongly under the bounded linear regularity property with exponent τ. Moreover, estimates of the convergence rate are also provided.

Theorem 3.1. *Let $0 < \tau \leq 1$. Suppose that the SFP (1.1) satisfies the bounded linear regularity property with exponent τ. Let $\{x_n\}$ be a sequence generated by Algorithm 1.1 such that $\sum_{n=0}^\infty \alpha_n\beta_n = \infty$ and $\{\beta_n\}$ satisfies (1.3) or (1.4) or (1.5). Then, $\{x_n\}$ converges strongly to a solution x^* of the SFP (1.1). Furthermore, there exist $\delta > 0$, $0 < q < 1$, and $N \in \mathbb{N}$ such that for each $n > N$,*

$$\|x_n - x^*\| \leq \begin{cases} 2\mathrm{d}_S(x_N) q^{\sum_{k=N}^{n-1} \alpha_k\beta_k}, & \tau = 1, \\ 2\left(\mathrm{d}_S^{2(1-\frac{1}{\tau})}(x_N) + \left(\frac{1}{\tau} - 1\right)\delta \sum_{k=N}^{n-1} \alpha_k\beta_k\right)^{-\frac{\tau}{2(1-\tau)}}, & 0 < \tau < 1. \end{cases} \tag{3.2}$$

Proof. Without loss of generality, we assume $x_n \notin S$ for all $n \geq 0$ (otherwise, the algorithm terminates after finite steps, and the conclusion holds trivially). Then, in view of Algorithm 1.1, we have $Ax_n \notin Q$ for all $n \geq 0$. Fix $z \in S$ and $n \in \mathbb{N}^*$. Let

$$\nabla_{x_n} := A^*(I - P_Q)Ax_n.$$

Then,

$$\|\nabla_{x_n}\| \leq \|A\| \mathrm{d}_Q(Ax_n). \tag{3.3}$$

By Lemma 2.1(ii) and the equivalence (3.1), we have

$$\langle x_n - z, \nabla_{x_n}\rangle = \langle A(x_n - z), (I - P_Q)Ax_n\rangle \geq \|(I - P_Q)Ax_n\|^2 = \mathrm{d}_Q^2(Ax_n). \tag{3.4}$$

Set

$$y_n := P_C(x_n - \beta_n\nabla_{x_n}), \quad \forall n \in \mathbb{N}^*.$$

By Lemma 2.1(i), the operator P_C is nonexpansive, so we obtain that

$$\begin{aligned} \|y_n - z\|^2 &= \|P_C(x_n - \beta_n\nabla_{x_n}) - z\|^2 \\ &\leq \|x_n - \beta_n\nabla_{x_n} - z\|^2 \\ &= \|x_n - z\|^2 - 2\beta_n\langle x_n - z, \nabla_{x_n}\rangle + \beta_n^2\|\nabla_{x_n}\|^2. \end{aligned}$$

Combining this with (3.4) yields that

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 - \beta_n\left(2 - \beta_n\frac{\|\nabla_{x_n}\|^2}{\mathrm{d}_Q^2(Ax_n)}\right)\mathrm{d}_Q^2(Ax_n). \tag{3.5}$$

Next, we show that there exists $M \in \mathbb{N}^*$, such that

$$\beta_n\|\nabla_{x_n}\|^2 \leq 2\mathrm{d}_Q^2(Ax_n), \quad \forall n \geq M. \tag{3.6}$$

In fact, if (1.3) or (1.5) holds, then there exist $\eta > 0$ and $M \in \mathbb{N}^*$, such that

$$\beta_n \le \eta < \frac{2}{\|A\|^2}, \quad \forall n \ge M. \tag{3.7}$$

Thus, (3.6) follows from (3.3). For assumption (1.4), it follows from the definition of β_n that

$$\beta_n \|\nabla_{x_n}\|^2 = \rho_n \mathrm{d}_Q^2(Ax_n) < 2\mathrm{d}_Q^2(Ax_n),$$

which gives (3.6). Therefore, it follows from (3.5) that

$$\|y_n - z\| \le \|x_n - z\|. \tag{3.8}$$

To proceed, noting that $x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n$, it follows from the convexity of $\|\cdot\|$ that

$$\begin{aligned} \|x_{n+1} - z\| &= \|(1-\alpha_n)(x_n - z) + \alpha_n(y_n - z)\| \\ &\le (1-\alpha_n)\|x_n - z\| + \alpha_n\|y_n - z\|. \end{aligned}$$

Thus, by (3.8), one has

$$\|x_{n+1} - z\| \le \|x_n - z\|, \quad \forall n \ge M. \tag{3.9}$$

Therefore, the sequence $\{\|x_n - z\|\}$ is bounded. Hence, there exists $r > 0$ such that $\{z\} \cup \{x_n\} \subseteq C \cap \overline{\mathbf{B}(0,r)}$. By assumption that the SFP (1.1) satisfies the bounded linear regularity property with exponent τ, it follows from Definition 2.3 that there exists $\gamma_r > 0$ such that

$$\gamma_r \mathrm{d}_S(x_n) \le \mathrm{d}_Q^\tau(Ax_n), \quad \forall n \ge 0.$$

This, together with (3.5), implies that

$$\|y_n - z\|^2 \le \|x_n - z\|^2 - \gamma_r^{\frac{2}{\tau}} \beta_n \left(2 - \beta_n \frac{\|\nabla_{x_n}\|^2}{\mathrm{d}_Q^2(Ax_n)}\right) \mathrm{d}_S^{\frac{2}{\tau}}(x_n), \quad \forall n \ge M.$$

Since $z \in S$ is arbitrary, one has

$$\mathrm{d}_S^2(y_n) \le \mathrm{d}_S^2(x_n) - \gamma_r^{\frac{2}{\tau}} \beta_n \left(2 - \beta_n \frac{\|\nabla_{x_n}\|^2}{\mathrm{d}_Q^2(Ax_n)}\right) \mathrm{d}_S^{\frac{2}{\tau}}(x_n), \quad \forall n \ge M. \tag{3.10}$$

Note by (3.3) that

$$2 - \beta_n \frac{\|\nabla_{x_n}\|^2}{\mathrm{d}_Q^2(Ax_n)} \ge 2 - \beta_n \|A\|^2.$$

Then

$$\liminf_{n \to +\infty} \left(2 - \beta_n \frac{\|\nabla_{x_n}\|^2}{\mathrm{d}_Q^2(Ax_n)}\right) > 0. \tag{3.11}$$

In fact, if assumption (1.3) or (1.5) is satisfied, (3.11) follows from (3.7), while if assumption (1.4) is satisfied, one has

$$2 - \beta_n \frac{\|\nabla_{x_n}\|^2}{\mathrm{d}_Q^2(Ax_n)} = 2 - \rho_n$$

and so (3.11) follows from (1.4). Hence, there exists $N \ge M$ such that

$$\delta := \inf_{n \ge N} \left\{ \gamma_r^{\frac{2}{\tau}} \left(2 - \beta_n \frac{\|\nabla_{x_n}\|^2}{\mathrm{d}_Q^2(Ax_n)}\right) \right\} > 0.$$

Thus it follows from (3.10) that

$$\mathrm{d}_S^2(y_n) \le \mathrm{d}_S^2(x_n) - \delta \beta_n \mathrm{d}_S^{\frac{2}{\tau}}(x_n), \quad \forall n \ge N. \tag{3.12}$$

Recalling that $x_{n+1} = (1-\alpha_n)x_n + \alpha_n y_n$, one checks by simple calculation that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|(1-\alpha_n)x_n + \alpha_n y_n - z\|^2 \\ &= (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|y_n - z)\|^2 - \alpha_n(1-\alpha_n)\|x_n - y_n\|^2.\end{aligned}$$

Since $\alpha_n \in [0,1]$, it follows that

$$\|x_{n+1} - z\|^2 \leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|y_n - z)\|^2$$

and so

$$\mathrm{d}_S^2(x_{n+1}) \leq (1-\alpha_n)\mathrm{d}_S^2(x_n) + \alpha_n \mathrm{d}_S^2(y_n)$$

because $z \in S$ is arbitrary. This, together with (3.12), implies that

$$\mathrm{d}_S^2(x_{n+1}) \leq \mathrm{d}_S^2(x_n)(1 - \delta\beta_n\alpha_n \mathrm{d}_S^{2(\frac{1}{\tau}-1)}(x_n)), \quad \forall n \geq N. \tag{3.13}$$

Thus, the following inequality holds for each $n \geq N$:

$$\mathrm{d}_S^2(x_{n+1}) \leq \begin{cases} \mathrm{d}_S^2(x_N)\prod_{k=N}^{n}(1-\delta\alpha_k\beta_k), & \tau = 1; \\ \left(\mathrm{d}_S^{2(1-\frac{1}{\tau})}(x_N) + \left(\frac{1}{\tau}-1\right)\delta\sum_{k=N}^{n}\alpha_k\beta_k\right)^{-\frac{\tau}{1-\tau}}, & 0 < \tau < 1. \end{cases} \tag{3.14}$$

In fact, for the case when $\tau = 1$, (3.14) follows directly from (3.13), while for the case when $0 < \tau < 1$, (3.14) is seen to hold by applying Lemma 2.2 to $\{\mathrm{d}_S^2(x_n)\}, \{\delta\alpha_k\beta_k\}, \frac{1}{\tau} - 1$ in place of $\{\mu_k\}, \{\alpha_k\}, p$. Fix $n > N$. By (3.9), $\{\|x_m - P_S(x_n)\|\}_{m\geq n}$ is monotonically decreasing and so

$$\|x_m - x_n\| \leq \|x_m - P_S(x_n)\| + \|x_n - P_S(x_n)\| \leq 2\|x_n - P_S(x_n)\| = 2\mathrm{d}_S(x_n). \tag{3.15}$$

Note that for any $0 \leq t < 1$, $\ln(1-t) \leq -t$. Therefore,

$$\prod_{k=N}^{n}\sqrt{1-\delta\alpha_k\beta_k} = \exp\left\{\frac{1}{2}\sum_{k=N}^{n}\ln(1-\delta\alpha_k\beta_k)\right\} \leq q^{\sum_{k=N}^{n}\alpha_k\beta_k}, \quad n \geq N,$$

where $q := e^{-\frac{\delta}{2}}$. Combining this with (3.15) and (3.14) yields that for all $m > n > N$,

$$\|x_m - x_n\| \leq \begin{cases} 2\mathrm{d}_S(x_N)q^{\sum_{k=N}^{n-1}\alpha_k\beta_k}, & \tau = 1, \\ 2\left(\mathrm{d}_S^{2(1-\frac{1}{\tau})}(x_N) + \left(\frac{1}{\tau}-1\right)\delta\sum_{k=N}^{n}\alpha_k\beta_k\right)^{-\frac{\tau}{2(1-\tau)}}, & 0 < \tau < 1. \end{cases} \tag{3.16}$$

Since $\sum_{n=1}^{\infty}\alpha_k\beta_k = \infty$ (due to assumption), $\{x_n\}$ is a Cauchy sequence and so converges to a point x^*. Hence, (3.2) is seen to hold by letting $m \to \infty$ in (3.16). Furthermore, it follows from (3.14) that $\lim_{n\to\infty}\mathrm{d}_S(x_n) = 0$ and so $\mathrm{d}_S(x^*) = 0$. As S is closed, one has $x^* \in S$, that is, x^* is a solution of the SFP (1.1). The proof is complete. □

The following theorem studies the convergence of a sequence generated by Algorithm 1.1 such that stepsizes are allowed to switch between (1.3) and (1.4) for different n. Under the assumption that $\{\alpha_n\}$ is bounded from 0, the strong convergence together with estimates of convergence rate of Algorithm 1.1 is obtained. In particular, if the SFP (1.1) satisfies the bounded linear regularity property, the algorithm converges linearly.

Theorem 3.2. *Let $0 < \tau \leq 1$. Suppose that the SFP (1.1) satisfies the bounded linear regularity property with exponent τ. Let $\{x_n\}$ be a sequence generated by Algorithm 1.1 such that β_n satisfies (1.3) or (1.4),* *and* $\{\alpha_n\} \subseteq (\alpha, 1]$ *form some* $0 < \alpha \leq 1$. *Then* $\{x_n\}$ *converges strongly to a solution* x^* *of the SFP* (1.1). *Furthermore, there exists* $\eta > 0$ *such that for all* $n \in \mathbb{N}^*$,

$$\|x_n - x^*\| \leq \begin{cases} 2(1-\eta)^{\frac{1}{2}n} \mathrm{d}_S(x_0), & \tau = 1, \\ 2\left(\mathrm{d}_S^{2(1-\frac{1}{\tau})}(x_0) + \eta\left(\frac{1}{\tau} - 1\right)n\right)^{-\frac{\tau}{2(1-\tau)}}, & 0 < \tau < 1. \end{cases} \tag{3.17}$$

In particular, if $\tau = 1$, *then* $\{x_n\}$ *converges linearly.*

Proof. We claim that

$$\beta_n \geq \min\left\{\sigma_1, \frac{\sigma_2}{\|A\|^2}\right\} > 0, \quad \forall n \in \mathbb{N}^*, \tag{3.18}$$

and

$$2 - \beta_n \frac{\|\nabla_{x_n}\|^2}{\mathrm{d}_Q^2(Ax_n)} \geq \min\{2 - \beta_n\|A\|^2, 2 - \rho_n\} \geq \min\{\sigma_1\|A\|^2, \sigma_2\} > 0, \quad \forall n \in \mathbb{N}^*. \tag{3.19}$$

In fact, for the case when β_n satisfies (1.3), $\beta_n \geq \sigma_1$, while for the case when β_n satisfies (1.4), one has $\beta_n \geq \frac{\rho_n}{\|A\|^2} \geq \frac{\sigma_2}{\|A\|^2}$; hence (3.18) is seen to hold. Additionally, for the case when β_n satisfies (1.3), it follows from (3.3) that

$$2 - \beta_n \frac{\|\nabla_{x_n}\|^2}{\mathrm{d}_Q^2(Ax_n)} \geq 2 - \beta_n\|A\|^2 > \sigma_1\|A\|^2,$$

while for the case when β_n satisfies (1.4), one has that

$$2 - \beta_n \frac{\|\nabla_{x_n}\|^2}{\mathrm{d}_Q^2(Ax_n)} \geq 2 - \rho_n > \sigma_2;$$

thus, (3.19) is checked. To proceed, for simplicity, write

$$\eta_1 := \min\left\{\sigma_1, \frac{\sigma_2}{\|A\|^2}\right\}, \quad \eta_2 := \min\left\{\sigma_1\|A\|^2, \sigma_2\right\}.$$

With similar arguments as done for (3.13), one checks that

$$\mathrm{d}_S^2(x_{n+1}) \leq \mathrm{d}_S^2(x_n)(1 - \gamma_r^{\frac{2}{\tau}} \eta_2 \beta_n \alpha_n \mathrm{d}_S^{2(\frac{1}{\tau}-1)}(x_n)), \quad \forall n \in \mathbb{N}^*. \tag{3.20}$$

Thus, it follows from (3.18), the definition of η_1 and the fact $\{\alpha_n\} \subseteq (\alpha, 1]$ that

$$\mathrm{d}_S^2(x_{n+1}) \leq \mathrm{d}_S^2(x_n)(1 - \gamma_r^{\frac{2}{\tau}} \eta_2 \eta_1 \alpha \mathrm{d}_S^{2(\frac{1}{\tau}-1)}(x_n)), \quad \forall n \in \mathbb{N}^*. \tag{3.21}$$

Set $\eta := \gamma_r^{\frac{2}{\tau}} \eta_2 \eta_1 \alpha$. Then, for each $n \in \mathbb{N}^*$,

$$\mathrm{d}_S^2(x_{n+1}) \leq \begin{cases} \mathrm{d}_S^2(x_0)(1-\eta)^{n+1}, & \tau = 1, \\ \left(\mathrm{d}_S^{2(1-\frac{1}{\tau})}(x_0) + \left(\frac{1}{\tau} - 1\right)\eta(n+1)\right)^{-\frac{\tau}{1-\tau}}, & 0 < \tau < 1. \end{cases} \tag{3.22}$$

Indeed, for the case when $\tau = 1$, (3.22) follows directly from (3.21), while for the case when $0 < \tau < 1$, (3.22) holds by applying Lemma 2.2 to $\{\mathrm{d}_S^2(x_n)\}, \{\eta\}, \frac{1}{\tau} - 1$ in place of $\{\mu_k\}, \{\alpha_k\}, p$. Then, with similar techniques as done for the proof of Theorem 3.1, one checks that the conclusions hold. The proof is complete. □

For the case when $\alpha_n = 1$ for each n, Algorithm 1.1 is reduced to the CQ algorithm (1.2). Therefore, Corollaries 3.3 and 3.4 follows directly from Theorems 3.1 and 3.2, respectively, where Corollary 3.3 extends the corresponding one in [25, Theorem 2.3]. Consider the assumption:

$$\lim_{n\to\infty} \beta_n = 0 \text{ and } \sum_{n=0}^{\infty} \beta_n = \infty. \tag{3.23}$$

Corollary 3.3. *Let $0 < \tau \leq 1$. Suppose that the SFP (1.1) satisfies the bounded linear regularity property with exponent τ. Let $\{x_n\}$ be a sequence generated by the CQ algorithm (1.2) such that $\{\beta_n\}$ satisfies (1.3) or (1.4) or (3.23). Then, $\{x_n\}$ converges strongly to a solution x^* of the SFP (1.1). Furthermore, there exist $\delta > 0$, $0 < q < 1$, and $N \in \mathbb{N}$ such that for each $n > N$,*

$$\|x_n - x^*\| \leq \begin{cases} 2\mathrm{d}_S(x_N) q^{\sum_{k=N}^{n-1} \beta_k}, & \tau = 1; \\ 2\left(\mathrm{d}_S^{2(1-\frac{1}{\tau})}(x_N) + \left(\frac{1}{\tau} - 1\right)\delta \sum_{k=N}^{n-1} \beta_k\right)^{-\frac{\tau}{2(1-\tau)}}, & 0 < \tau < 1. \end{cases}$$

In particular, if $\tau = 1$ and (1.3) or (1.4) holds, then $\{x_n\}$ converges linearly.

Corollary 3.4. *Let $0 < \tau \leq 1$. Suppose that the SFP (1.1) satisfies the bounded linear regularity property with exponent τ. Let $\{x_n\}$ be a sequence generated by the CQ algorithm (1.2) such that β_n satisfies (1.3) or (1.4). Then $\{x_n\}$ converges strongly to a solution x^* of the SFP (1.1). Furthermore, there exists $\eta > 0$ such that for all $n \in \mathbb{N}^*$,*

$$\|x_n - x^*\| \leq \begin{cases} 2(1-\eta)^{\frac{1}{2}n}\mathrm{d}_S(x_0), & \tau = 1, \\ 2\left(\mathrm{d}_S^{2(1-\frac{1}{\tau})}(x_0) + \eta\left(\frac{1}{\tau} - 1\right) n\right)^{-\frac{\tau}{2(1-\tau)}}, & 0 < \tau < 1. \end{cases}$$

In particular, if $\tau = 1$, then $\{x_n\}$ converges linearly.

4. Numerical Experiments

In this section, we present some numerical experiments to demonstrate the effectiveness of Algorithm 1.1. All the tests are implemented in R(4.4.3) on a personal computer with AMD R7 7735H, Radeon Graphics 3.20 GHz and RAM 16.00 GB.

We consider the compressed sensing problem described in [8], which can be approximated by a linear system of the form $b = Ax + e$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ are known, $e \in \mathbb{R}^m$ is an arbitrary and unknown vector of errors, and $x \in \mathbb{R}^n$ is a variable to be estimated. The sparsity of x is measured by the ℓ_1-norm defined by $\|x\|_1 := \sum_{i=1}^n |x_i|$. Let $t \geq 0$ be a constant and $\varepsilon := \|e\|$. Write

$$C := \{x \in \mathbb{R}^n \mid \|x\|_1 \leq t\} \quad \text{and} \quad Q := \{y \in \mathbb{R}^m \mid \|y - b\|_2 \leq \varepsilon\}. \tag{4.1}$$

Thus, the compressed sensing problem can be viewed as the SFP (1.1) with $H_1 = \mathbb{R}^n$ and $H_2 = \mathbb{R}^m$. Then we can check that the SFP (1.1) satisfies the bounded regularity property with exponent $\frac{1}{2}$; see Remark 4.1.

Remark 4.1. The SFP (1.1) (with C and Q given by (4.1)) satisfies the bounded linear regularity property with exponent $\frac{1}{2}$. Indeed, let

$$c(x) := \max_{\alpha\in\Lambda}\{\alpha^T x - t\}, \quad \forall x \in \mathbb{R}^n,$$

where $\Lambda := \{\alpha \in \mathbb{R}^n : \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T, \alpha_i \in \{1, -1\}, i = 1, 2, \ldots, n\}$, and let

$$q(y) := \|y - b\|_2^2 - \varepsilon^2, \quad \forall y \in \mathbb{R}^m.$$

Then the compressed sensing problem can be viewed as the SFP (1)-(2) in [29]. As pointed out in [29, Remark 4.1], the SFP (1)-(2) satisfies the bounded error bound condition with exponent $\frac{1}{2}$, that is, for any $r > 0$ with $S \cap \overline{\mathbf{B}(0, r)} \neq \emptyset$, there exists $\widetilde{\gamma}_r > 0$ such that

$$\widetilde{\gamma}_r \mathrm{d}_S^2(x) \leq \max\{[c(x)]_+, [q(Ax)]_+\}, \quad \forall x \in \overline{\mathbf{B}(0, r)}, \tag{4.2}$$

where $a_+ = \max\{a, 0\}$. Fix $r > 0$ with $S \cap \overline{\mathbf{B}(0, r)} \neq \emptyset$, and let $M_r := \sup_{x\in C\cap\overline{\mathbf{B}(0,r)}}\{\|Ax - b\|_2 + \varepsilon\}$. Recall that $C = \{x \in \mathbb{R}^n \mid c(x) \leq 0\}$ and $Q = \{y \in \mathbb{R}^m \mid q(y) \leq 0\}$. Then, we have

$$\max\{[c(x)]_+, [q(Ax)]_+\} = [q(Ax)]_+ \leq M_r \mathrm{d}_Q(Ax), \quad \forall x \in C \cap \overline{\mathbf{B}(0, r)}. \tag{4.3}$$

In fact, given $x \in C \cap \overline{\mathbf{B}(0,r)}$, if $q(Ax) \le 0$, (4.3) is trivial; otherwise, by definition, one has

$$\max\{[c(x)]_+, [q(Ax)]_+\} = [q(Ax)]_+ = q(Ax) = \|Ax-b\|_2^2 - \varepsilon^2 \le M_r(\|Ax-b\|_2 - \varepsilon) = M_r \mathrm{d}_Q(Ax)$$

and so (4.3) is checked. Thus, it follows from (4.2) and (4.3) that

$$\widetilde{\gamma}_r \mathrm{d}_S^2(x) \le M_r \mathrm{d}_Q(Ax), \quad \forall x \in C \cap \overline{\mathbf{B}(0,r)}.$$

Therefore, (2.1) holds with $\tau = \frac{1}{2}$ and $\gamma_r = \sqrt{\frac{\widetilde{\gamma}_r}{M_r}}$.

We carry out four experiments to compare the convergence results between Algorithm 1.1, the relaxed CQ algorithm [29] and the RSSEA algorithm [8]. In each experiment, the simulated data are generated via the standard process of compressive sensing. In detail, we randomly generate an independent and identically distributed Gaussian ensemble $A \in \mathbb{R}^{m\times n}$ satisfying $A^\top A = I$. The true sparse solution $\bar{x} \in \mathbb{R}^n$ has $s \in \mathbb{N}$ nonzero elements drawn independently from a Gaussian distribution, and t is obtained by $t = \|\bar{x}\|_1$. The observation vector b is generated via $b = A\bar{x}$. The problem size is set as $m = 256$ and $n = 1024$, with initial point $x_0 = 0$ and error $\varepsilon = 10^{-6}$. To evaluate the performance of algorithms, we compute the total violation by

$$\text{Total violation} := [\|x\|_1 - t]_+ + [\|Ax-b\|_2 - \epsilon]_+.$$

The first experiment demonstrates the convergence results of Algorithm 1.1 with fixed stepsizes $\beta_n \equiv 1$ and $\alpha_n \equiv \frac{1}{4}$. We conduct 100 trials with randomly simulated data to show the convergence property of Algorithm 1.1. See Figure 1.

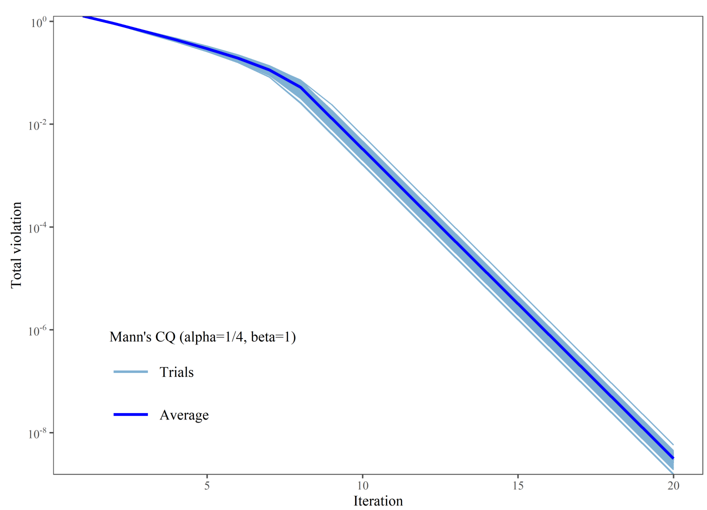
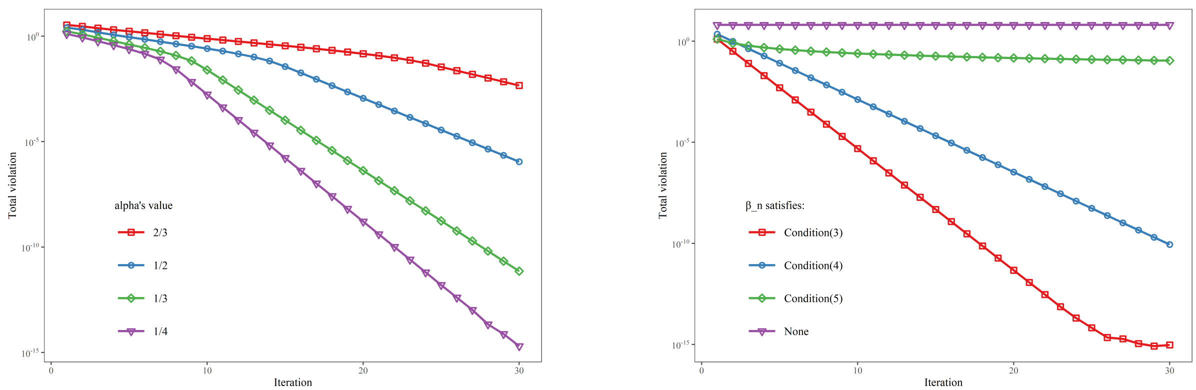


FIGURE 1. Convergence results of Algorithm 1.1 after 100 iterations

The second experiment shows that the convergence results of Algorithm 1.1 with different values of α_n or different stepsizes β_n. In Figure 2(a), we choose the fixed stepsize $\beta_n \equiv 1$ and different values of $\alpha_n \in \{\frac{2}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$, while we consider in Figure 2(b) the fixed parameter $\alpha_n \equiv \frac{1}{4}$ and different stepsize β_n satisfying one of the following conditions:

- Condition (1.3): $\sigma_1 = \frac{1}{\|A\|^2}$ and $\beta_n = \frac{1}{\|A\|^2}$.
- Condition (1.4): $\sigma_2 = \frac{1}{2}$, $\rho_n = 1$, and $\beta_n = \frac{\rho_n \|(I-P_Q)Ax_n\|^2}{\|A^*(I-P_Q)Ax_n\|^2}$.
- Condition (1.5): $\beta_n = \frac{1}{n}$.
- None of the above conditions: $\beta_n \equiv 3$.



(a) Different values of α_n

(b) Different step size β_n

FIGURE 2. Convergence results with different values of α_n or step size β_n

The last experiment compares the convergence performance of Algorithm 1.1, the relaxed CQ algorithm, and the RSSEA, where we choose $\alpha_n \equiv \frac{1}{20}$ and $\beta_n \equiv 0.9$ for Algorithm 1.1, $\beta_n \equiv 0.9$ for the relaxed CQ algorithm, and $\beta_n \equiv 0.9$ for the RSSEA, respectively. Compared to the other two algorithms, Algorithm 1.1 has a faster convergence speed; see Figure 3.

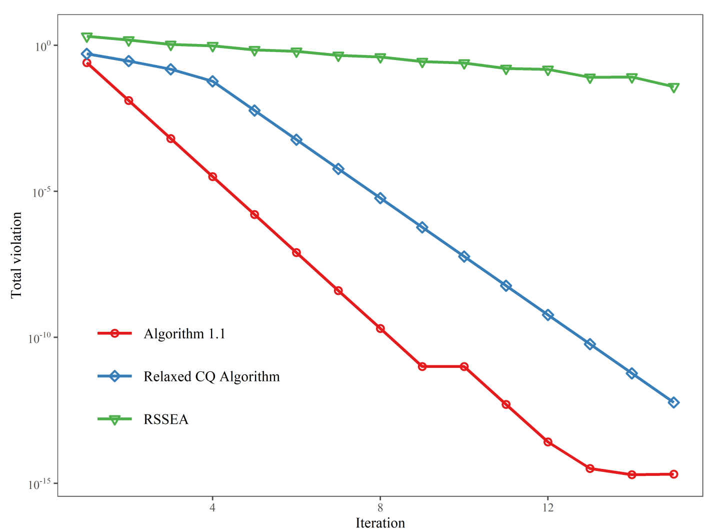


FIGURE 3. Convergence performance among different algorithms

STATEMENTS AND DECLARATIONS

No potential conflict of interest was reported by the authors.

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