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PARAMETRIC SUMMABILITY AND ITS APPLICATIONS TO MAXIMIZING OF THE SUMMABILITY DOMAIN

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Abstract. In this paper, we study parametric summability based on parameterized double sequences of complex constants as it is defined in "Linear Operators, General Theory" by N. Dunford and J. T. Schwartz. We define "power double sequences" or infinite "power matrices" as certain generalizations of double sequences and power series. We show that the parameter dependence of the summability of power double sequences is similar to convergence of power series and we introduce the radius of summability. That opens a way to maximize the summability domain using the radius of summability. While others do investigate "power matrices," their definitions, as far as we were able to find, differ from ours. Using our approach, we find new summability results for double sequences of constants in the case of power double sequences. We will give some applications to both standard summability theory and analytic functions. In section 7, we provide some examples to demonstrate the main results of this paper obtained in sections 5 and 6. Finally, to conclude this paper, in the last section, we give some ideas related to parametric summability for further study.

Keywords. Parametric summability, General power matrices, Silverman-Toeplitz theorem, Radius of summability.

1. Introduction

The practical need to improve convergence gave the impulse to study sequence transformations already in 17th century and resulted in the creation of summability theory at the end of 19th century. And the summability theory has been developed to be an important branch in the theory of analysis. It has been developed and studied for a long-time history by many researchers (see [1, 3, 25, 7, 10, 12, 13, 14, 22, 26]). The summability theory has been applied to many subjects in the theory of analysis (see [13, 14, 15, 22, 26]). The theme of the summability theory concentrates at the topics of convergent or divergent of sequences and series (see [13, 14, 15, 18, 23]). Before the invention of computers, mainly linear sequence transformations were studied. Approaches based on classical analysis culminated when [13] was published. After those modern approaches based on functional analysis appeared. For a comprehensive review of classical and modern methods in summability (see [3]). From practical point of view, regular linear transformations are in general at most moderately powerful in improving convergence, and the popularity of most linear transformations has declined considerably in recent years. It seems, however, that the limiting factor is regularity not linearity. Since there are different reasons for transforming one sequence into another, then many researchers have studied infinite matrices and summability methods for double series sequence spaces (see [4, 6, 11, 12, 16]). Recently also new powerful non-linear sequence transformations attracted research and applications. This is discussed in a nice historical review [25]. One more step further, when a random variable involves the convergence of

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a series, the statistical summability of double sequences or series has been rapidly developed by many researches (see [2, 5, 8, 9, 10, 17, 19, 20, 21, 24]).

In this paper, we contribute to a new type of summability methods of parametric double sequences, which we call a parametric summability.

This paper is organized as follows. In section 2, we review some concepts in the summability theory and Silverman-Toeplitz Theorem. This is an important theorem in the summability theory, which will be used in this paper; In sections 3 and 4 we introduce parametric double sequences; In section 5, mainly by using the Silverman-Toeplitz theorem, we prove some summability results of parametric double sequences; In section 6, we study the radius of Summability parametric double sequences and provide some applications; In section 7, we give some applications of summability of parametric double sequences to analytic functions and provided some examples.

2. Preliminaries

Let $A = \{a_{ij}\},\ i = 1, 2, \dots,\ j = 0, 1, 2, \dots,$ be a double sequence of complex constants, that is,

$$A = \begin{pmatrix} a_{10} & a_{11} & a_{12} & \dots \\ a_{20} & a_{21} & a_{22} & \dots \\ a_{30} & a_{31} & a_{32} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

Let Δ be the set of all double sequences of complex constants and c be the space of all convergent sequences of scalars.

Definition 2.1. Suppose that a double sequence $\{a_{ij}\}$ defines a linear transformation T of c onto itself by means of formula

$$T[s_1, s_2, \dots] = [t_1, t_2, \dots] = [\sum_{j=0}^{\infty} a_{1j} s_j, \sum_{j=0}^{\infty} a_{2j} s_j,\ \dots].$$

If T preserves limits of sequences (i.e. if $\lim_{i\to\infty} t_i = \lim_{j\to\infty} s_j$ for every sequence (or vector) $[s_j] \in c$), then the double sequence (or matrix) $A = \{a_{ij}\}$ is said to define a *regular method of summability*.

Lemma 2.2. *A defines a bounded linear map of c into c, if and only if the following three conditions hold:*

(1) $\underset{1\leq i<\infty}{lub} \sum_{j=0}^{\infty} |a_{ij}| = M < \infty$;
(2) $\lim_{i\to\infty} a_{ij}$ *exists for j = 0, 1, 2, ... ;*
(3) $\lim_{i\to\infty} \sum_{j=0}^{\infty} a_{ij}$ *exists.*

Once A satisfies the above three conditions, for any column vector $s = [s_0, s_1, s_2,\ \dots] \in c$, the bounded linear map, A, of c into c is defined by

$$A(s)_i = a_{i0} \lim_{j\to\infty} s_j + \sum_{j=1}^{\infty} a_{ij} s_j, \text{ for } i = 1, 2, \dots,$$

where $A(s)$ = $[A(s)_1, A(s)_2, A(s)_3, \dots]$.

Let B_c denote the space of all linear bounded maps of c into c. From Lemma 2.2, if $A \in B_c$, then it has norm

$$|A| = M.$$

Theorem 2.3. ***(Silverman-Toeplitz)*** *A defines a regular method of summability, if and only if the following three conditions hold:*

(1) $\underset{1\leq i<\infty}{lub}\sum_{j=0}^{\infty}|a_{ij}| = M < \infty$;
(2) $\lim_{i\to\infty} a_{ij} = 0$ *for* $j = 0, 1, 2, \ldots$;
(3) $\lim_{i\to\infty}\sum_{j=0}^{\infty} a_{ij} = 1.$

It is clear that if A defines a regular method of summability, then $A \in B_c$. One can find all, the Definition, the Lemma, and Silverman-Toeplitz Theorem in [7].

One more step further, let $A(z) = \{f_{ij}(z)\},\ i = 1, 2, \ldots,\ j = 0, 1, 2, \ldots$, be a double sequence of functions with same domain D, which is a subset of complex numbers, that is,

$$A(z) = \begin{pmatrix} f_{10}(z) & f_{11}(z) & f_{12}(z) & \ldots \\ f_{20}(z) & f_{21}(z) & f_{22}(z) & \ldots \\ f_{30}(z) & f_{31}(z) & f_{32}(z) & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

It is clear that for any input $a \in D$, the output of $A(a) \in \Delta$. For any given $a \in D$ we have the following Lemma and Theorem.

Lemma 2.4. *Let* $a \in D$*, then A(a) defines a bounded linear map of c into c, if and only if the following three conditions hold:*

(1) $\underset{1\leq i<\infty}{sub}\sum_{j=0}^{\infty}|f_{ij}(a)| = M < \infty$;
(2) $\lim_{i\to\infty} f_{ij}(a)$ *exists for j = 0, 1, 2, ... ;*
(3) $\lim_{i\to\infty}\sum_{j=0}^{\infty} f_{ij}(a)$ *exists.*

Once A(*a*) satisfies the above three conditions, for any column vector $s = [s_1, s_2, s_3, \ \ldots] \in c$, the bounded linear map, A(*a*), of *c* into *c* is defined by

$$A(a)(s)_i = f_{i0}(a)\lim_{j\to\infty} s_j + \sum_{j=1}^{\infty} f_{ij}(a)s_j, \text{ for } i = 1, 2, \ldots,$$

where A(*a*)(s) = [A(*a*)$(s)_1$, A(*a*)$(s)_2$, A(*a*)$(s)_3$, $\ldots$].

Let B_c denote the space of all linear bounded maps of *c* into *c*. From Lemma 2.2, if A(*a*) $\in B_c$, then it has norm

$$|A(a)| = M.$$

Theorem 2.5. ***(Silverman-Toeplitz)*** *Let* $a \in D$*, then A(a) defines a regular method of summability, if and only if the following three conditions hold:*

(1) $\underset{1\leq i<\infty}{lub}\sum_{j=0}^{\infty}|f_{ij}(a)| = M < \infty$;
(2) $\lim_{i\to\infty} f_{ij}(a) = 0$ *for* $j = 0, 1, 2, \ldots$;
(3) $\lim_{i\to\infty}\sum_{j=0}^{\infty} f_{ij}(a) = 1.$

Again, it is clear that if A(*a*) defines a regular method of summability, then A(*a*) $\in B_c$.

3. Power Matrices

In this section, we consider special matrices of functions, in which every function $f_{ij}(z)$ is a monomial of z. Let $A = \{ a_{ij} \}$, $i = 1, 2, \dots , j = 0, 1, 2, \dots$, be a double sequence of complex constants.

The column power matrix induced by A is defined as

$$P_A^C(z) = \{a_{ij}z^i\}, i = 1, 2, \dots, j = 0, 1, 2, \dots, \text{ that is,}$$

$$P_A^C(z) = \begin{pmatrix} a_{10}z & a_{11}z & a_{12}z & \dots \\ a_{20}z^2 & a_{21}z^2 & a_{22}z^2 & \dots \\ a_{30}z^3 & a_{31}z^3 & a_{32}z^3 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

The row power matrix induced by A is defined as

$$P_A^R(z) = \{a_{ij}z^j\}, i = 1, 2, \dots, j = 0, 1, 2, \dots, \text{ that is,}$$

$$P_A^R(z) = \begin{pmatrix} a_{10} & a_{11}z & a_{12}z^2 & \dots \\ a_{20} & a_{21}z & a_{22}z^2 & \dots \\ a_{30} & a_{31}z & a_{32}z^2 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

The double power matrix induced from A is defined as

$$P_A(z) = \{a_{ij}z^{i+j}\}, i = 1, 2, \dots, j = 0, 1, 2, \dots, \text{ that is,}$$

$$P_A(z) = \begin{pmatrix} a_{10}z & a_{11}z^2 & a_{12}z^3 & \dots \\ a_{20}z^2 & a_{21}z^3 & a_{22}z^4 & \dots \\ a_{30}z^3 & a_{31}z^4 & a_{32}z^5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

We will immediately generalize these definitions in the following section.

4. General Power Matrices

We will define power matrices of the first type now.

Definition 4.1. Let $A = \{a_{ij}\}$, $i = 0, 1, 2, \dots$, $j = 0, 1, 2, \dots$, be a double sequence of complex constants. Let $g(z) = \sum_{i=0}^{\infty} g_i z^i$ be a complex power series. Denote its radius of convergence as R_g. The *column power matrix* induced by A and associated with $g(z)$ is defined as
$P_{A;g}^C(z) = \{a_{ij}g_i z^i\}$, $i = 0, 1, 2, \dots$, $j = 0, 1, 2, \dots$, that is,

$$P_{A;g}^C(z) = \begin{pmatrix} a_{00}g_0 & a_{01}g_0 & a_{02}g_0 & \dots \\ a_{10}g_1z & a_{11}g_1z & a_{12}g_1z & \dots \\ a_{20}g_2z^2 & a_{21}g_2z^2 & a_{22}g_2z^2 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Definition 4.2. Let $A = \{a_{ij}\}$, $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$, be a double sequence of complex constants. Let $h(z) = \sum_{j=0}^{\infty} h_j z^j$ be a complex power series. Denote its radius of convergence as R_h. The *row power matrix* induced by A and associated with $h(z)$ is defined as
$P^R_{A;h}(z) = \{a_{ij} h_j z^j\}$, $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$:, that is,

$$P^R_{A;h}(z) = \begin{pmatrix} a_{00}h_0 & a_{01}h_1 z & a_{02}h_2 z^2 & \ldots \\ a_{10}h_0 & a_{11}h_1 z & a_{12}h_2 z^2 & \ldots \\ a_{20}h_0 & a_{21}h_1 z & a_{22}h_2 z^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

We will generalize the double power matrix $P_A(z)$ now.

Definition 4.3. Let $A = \{a_{ij}\}$, $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$, be a double sequence of complex constants. Let $g(z) = \sum_{i=0}^{\infty} g_i z^i$ be a complex power series. Denote its radius of convergence respectively as r_g. The *power double sequence* of *second* type induced by A and associated with $g(z)$ is defined as
$P_{A;g}(z) = \{a_{ij} g_{i+j} z^{i+j}\}$, $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$:

$$P_{A;g}(z) = \begin{pmatrix} a_{00}g_0 & a_{01}g_1 z & a_{02}g_2 z^2 & \ldots \\ a_{10}g_1 z & a_{11}g_2 z^2 & a_{12}g_3 z^3 & \ldots \\ a_{20}g_2 z^2 & a_{21}g_3 z^3 & a_{22}g_4 z^4 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

Definition 4.4. Let $A = \{a_{ij}\}$, $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$, be a double sequence of complex constants. Let $g(z) = \sum_{i=0}^{\infty} g_i z^i$ and $h(z) = \sum_{j=0}^{\infty} h_j z^j$ be two complex power series. Denote their radius of convergence respectively as r_g and r_h. The *power double sequence* of *third* type induced by A and associated with $g(z)$ and $h(z)$ is defined as
$P_{A;g,h}(z) = \{a_{ij} g_i h_j z^{i+j}\}$, $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$:

$$P_{A;g,h}(z) = \begin{pmatrix} a_{00}g_0h_0 & a_{01}g_0h_1 z & a_{02}g_0h_2 z^2 & \ldots \\ a_{10}g_1h_0 z & a_{11}g_1h_1 z^2 & a_{12}g_1h_2 z^3 & \ldots \\ a_{20}g_2h_0 z^2 & a_{21}g_2h_1 z^3 & a_{22}g_2h_2 z^4 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

Remark 4.5. More general definition would consider $[g_i]$ and $[h_j]$ to be two arbitrary number sequences.

5. Summability Results

For power double sequences of first type we have:

Proposition 5.1. *Let the double sequence of complex constants $\{a_{ij}\}$, $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$, be a regular method of summability and $g(z) = \sum_{i=0}^{\infty} g_i z^i$ be a complex power series. Then the following two conditions are equivalent for any complex number z:*
(i) $\lim_{i\to\infty} g_i z^i = 1$.
(ii) *The power double sequence of first type* $\left\{ \left(P^C_{A;g}(z) \right)_{ij} \right\}$ *is a regular method of summability.*

Proof. (i) implies (ii) is a straightforward verification of the three conditions of Silverman-Toeplitz Theorem (Theorem 2.3). (ii) implies (i). The condition (3) of Silverman-Toeplitz Theorem for $P^C_{A;g}$ and for A gives (i). □

Proposition 5.2. *Let the double sequence of complex constants* $\{a_{ij}\}$, $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$, *be a regular method of summability and* $h(z) = \sum_{j=0}^{\infty} h_j z^j$ *be a complex power series. Then the following two conditions are equivalent for any complex number* z, *which the sequence* $[h_i z^i]$ *is convergent for:*

(i) $\lim_{j\to\infty} h_j z^j = 1$.

(ii) *The power double sequence of first type* $\left\{ \left(P^R_{A;h}(z) \right)_{ij} \right\}$ *is a regular method of summability.*

Proof. (i) implies (ii) is a straightforward verification of the three conditions of Silverman-Toeplitz Theorem. (ii) implies (i). The condition (3) of Silverman-Toeplitz Theorem for $P^R_{A;h}$ and for A along with the convergence of the sequence $[h_i z^i]$ gives (i) (see more details in the proof of (ii) implies (i) in the Theorem 5.3 below). □

For power double sequences of second type we have:

Theorem 5.3. *Let double sequence of complex constants* $\{a_{ij}\}$, $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$, *be a regular method of summability and* $g(z) = \sum_{i=0}^{\infty} g_i z^i$ *be a complex power series. Then the following two conditions are equivalent for any complex number* z, *which the sequence* $[g_i z^i]$ *is convergent for:*

(i) $\lim_{i\to\infty} g_i z^i = 1$.

(ii) *The power double sequence of second type* $\left\{ \left(P_{A;g}(z) \right)_{ij} \right\}$ *is a regular method of summability.*

Proof. Let's show first that (ii) implies (i). From (ii) and condition (3) of Silverman-Toeplitz theorem, we have
$\lim_{i\to\infty} \sum_{j=0}^{\infty} a_{ij} g_{i+j} z^{i+j} = 1$. Set $k = i + j$. This changes into $\lim_{i\to\infty} \sum_{k=i}^{\infty} a_{i,k-i} g_k z^k = 1$. Set $b_{ik} = a_{i,k-i}$ for $i \le k$ and zero otherwise. Observe $\{b_{ik}\}$ is also a regular method of summability. Then by Silverman-Toeplitz theorem the limits of convergent sequences are preserved: $\lim_{k\to\infty} g_k z^k = \lim_{i\to\infty} \sum_{k=0}^{\infty} b_{i,k} g_k z^k$. But the right hand side is equal to $\lim_{i\to\infty} \sum_{k=i}^{\infty} a_{i,k-i} g_k z^k = 1$ and (i) immediately follows.

Now we show that (i) implies (ii). We will use Silverman-Toeplitz theorem again and we need to prove its three conditions:

1. $\sup_{0\le i<\infty} \sum_{j=0}^{\infty} \left| \left(P_{A;g}(z) \right)_{ij} \right| = \sup_{0\le i<\infty} \sum_{j=0}^{\infty} |a_{ij} g_{i+j} z^{i+j}|$ by the definition of $P_{A;g}$.

Set $k = i + j$. Then the above supremum is

$$\sup_{0\le i<\infty} \sum_{k=i}^{\infty} |a_{i,k-i} g_k z^k| \le \sup_{0\le i<\infty} \sum_{k=i}^{\infty} |a_{i,k-i}| \cdot \sup_{0\le k<\infty} |g_k z^k| < \infty$$

since the first supremum is finite by condition (1) of $\{a_{ij}\}$ being a regular method of summability and the second one by the existence of the limit in (i).

2. $\lim_{i\to\infty} \left(P_{A;g}(z) \right)_{ij} = \lim_{i\to\infty} a_{ij} g_{i+j} z^{i+j}$ for $j = 0, 1, 2, \ldots$ by the definition.

Set $k = i + j$. Then the absolute value of the above limit is $\left|\lim_{k\to\infty} a_{k-j,j} g_k z^k\right| \leq \lim_{k\to\infty} |a_{k-j,j}| \cdot \lim_{k\to\infty} \left|g_k z^k\right| = 0$ since the first limit is zero by condition (2) of $\{a_{ij}\}$ being a regular method of summability and the second limit is 1 by (*i*).

3. $\lim_{i\to\infty} \sum_{j=0}^{\infty} (P_{A;g}(z))_{ij} = \lim_{i\to\infty} \sum_{j=0}^{\infty} a_{ij} g_{i+j} z^{i+j}$ by the definition.

Set $k = i + j$. Starting with (*i*) following the fist part of this proof ((*ii*) implies (*i*)) backwards we have $1 = \lim_{k\to\infty} g_k z^k = \lim_{i\to\infty} \sum_{k=0}^{\infty} b_{i,k} g_k z^k = \lim_{i\to\infty} \sum_{k=i}^{\infty} a_{i,k-i} g_k z^k = \lim_{i\to\infty} \sum_{j=0}^{\infty} a_{ij} g_{i+j} z^{i+j}$.

And this finishes the proof that $\left\{(P_{A;g}(z))_{ij}\right\}$ is a regular method of summability by Silverman-Toeplitz theorem. □

Remark 5.4. The condition (*i*) in Theorem 5.3 implies $|z| = r_g$, where r_g is the radius of convergence. The requirement in the Theorem 5.3 that the sequence $[g_i z^i]$ must be convergent seems to be too restrictive but the condition (*ii*) does not guarantee its convergence. There are examples of non-convergent sequences $[g_i z^i]$ (for both bounded and unbounded case) and regular methods of summability that map these sequences to convergent ones. Then by choosing $z = 1$ one has a counterexample for each case.

Corollary 5.5. *1. From the proof of the Theorem 5.3 it is clear that for $|z| < r_g$, conditions 1. and 2. hold but the limit in condition 3. is zero and we don't get a regular method of summability in that case.*

For power double sequences of third type we have:

Theorem 5.6. *Let double sequence of complex constants $\{a_{ij}\}$, $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$, be a regular method of summability, and $g(z) = \sum_{i=0}^{\infty} g_i z^i$ and $h(z) = \sum_{j=0}^{\infty} h_j z^j$ be two complex power series. If $\lim_{i\to\infty} g_i z^i$ and $\lim_{j\to\infty} h_j z^j$ exist then the following two conditions are equivalent for any such complex number z:*

(i) $\lim_{i\to\infty} g_i z^i \cdot \lim_{j\to\infty} h_j z^j = 1.$

(ii) *The power double sequence of second type $\left\{(P_{A;g,h}(z))_{ij}\right\}$ is a regular method of summability.*

Proof. The main structure of this proof is similar to the one of the Theorem 5.3 Let's show first that (*ii*) implies (*i*). From (*ii*) and condition (3) of Silverman-Toeplitz theorem, we have $1 = \lim_{i\to\infty} \sum_{j=0}^{\infty} a_{ij} g_i h_j z^{i+j}$. This equals to $\lim_{i\to\infty} g_i z^i \cdot \lim_{i\to\infty} \sum_{j=0}^{\infty} a_{ij} h_j z^j = \lim_{i\to\infty} g_i z^i \cdot \lim_{j\to\infty} h_j z^j$, where we also used that $\{a_{ij}\}$ as a linear operator preserves limits by Silverman-Toeplitz theorem.

Now we show that (*i*) implies (*ii*). We will use again Silverman-Toeplitz theorem and need to prove its three conditions:

1. $\sup_{0\leq i<\infty} \sum_{j=0}^{\infty} \left|(P_{A;g,h}(z))_{ij}\right| = \sup_{0\leq i<\infty} \sum_{j=0}^{\infty} \left|a_{ij} g_i h_j z^{i+j}\right|$ by the definition of $P_{A;g,h}$.

The above supremum equals to

$$\sup_{0\leq i<\infty} \left|g_i z^i\right| \cdot \sum_{j=0}^{\infty} \left|a_{ij} h_j z^j\right| \leq \sup_{0\leq i<\infty} \sum_{j=0}^{\infty} |a_{i,j}| \cdot \sup_{0\leq i<\infty} \left|g_i z^i\right| \cdot \sup_{0\leq j<\infty} \left|h_j z^j\right| < \infty$$

since the first supremum is finite by condition (1) of $\{a_{ij}\}$ being a regular method of summability and the other two are finite by the existence of the limit in (*i*).

2. $\lim_{i\to\infty}(P_{A;g,h}(z))_{ij} = \lim_{i\to\infty} a_{ij}g_ih_jz^{i+j}$ for $j = 0, 1, 2, \ldots$ by the definition.

The absolute value of the above limit is $\left|h_jz^j \cdot \lim_{i\to\infty} a_{ij}g_iz^i\right| \leq \sup_{0\leq i<\infty} |a_{i,j}| \cdot \left|h_jz^j \cdot \lim_{i\to\infty} g_iz^i\right| = 0$ for $j = 0, 1, 2, \ldots$
since the first limit is zero by condition (2) of $\{a_{ij}\}$ being a regular method of summability and the second limit is finite by (*i*).

3. $\lim_{i\to\infty}\sum_{j=0}^{\infty}(P_{A;g,h}(z))_{ij} = \lim_{i\to\infty}\sum_{j=0}^{\infty} a_{ij}g_ih_jz^{i+j}$ by the definition.

Starting with (*i*) following the proof of necessary condition backwards we have:
$1 = \lim_{i\to\infty} g_iz^i \cdot \lim_{j\to\infty} h_jz^j = \lim_{i\to\infty} g_iz^i \cdot \lim_{i\to\infty}\sum_{j=0}^{\infty} a_{ij}h_jz^j = \lim_{i\to\infty}\sum_{j=0}^{\infty} a_{ij}g_ih_jz^{i+j}$.

And this finishes the proof that $\left\{(P_{A;g,h}(z))_{ij}\right\}$ is a regular method of summability by Silverman-Toeplitz theorem. □

Remark 5.7. (*i*) implies $|z| = r_g = r_h$, where r_g and r_h are the radii of convergence of the respective series. The requirement in the Theorem 5.6 that both sequences $\{g_iz^i\}$ and $\{h_iz^i\}$ must be convergent seems to be too restrictive but the condition (*ii*) does not guarantee their convergence. There are examples of non-convergent sequences $\{g_iz^i\}$ and $\{h_iz^i\}$ and regular methods of summability that map these sequences to convergent ones. Then again by choosing $z = 1$ one has a counterexample for each case.

Corollary 5.8. *2. From the proof it is clear that for* $|z| < r_g$ *and* $|z| < r_h$*, conditions 1. and 2. hold but the limit in 3. is zero and we don't get a regular method of summability in that case.*

6. Radius of Summability and its Applications

Assume $A \in B_c$ now, but not necessarily a regular method of summability. It is clear that $P^R_{A;h}(0) \in B_c$ and $P^R_{A;h}(1) \in B_c$. Also $A = P^C_{A;g}(1) \in B_c$. On the other hand, for a given z we can ask: Does $P^C_{A;g}(z) \in B_c$ or $P^R_{A;h}(z) \in B_c$ hold? And because the conditions (1) and (2) clearly hold this is equivalent to: Does

$$\lim_{i\to\infty}\sum_{j=0}^{\infty} a_{ij}g_iz^i \quad \text{or} \quad \lim_{i\to\infty}\sum_{j=0}^{\infty} a_{ij}h_jz^j \quad \text{exist?}$$

The next propositions provide an answer for these two kinds of power matrices. For power double sequences of first type we have:

Proposition 6.1. *Let* $A = \{\ a_{ij}\ \}$*, i = 1, 2, … , j = 0, 1, 2, … , be a double sequence of complex scalars satisfying* $A \in B_c$*. If* $P^C_{A;g}(a) \in B_c$*, for some* $a \neq 0$*, then* $P^C_{A;g}(z) \in B_c$*, for all* z *satisfying* $|z| < |a|$*.*

Proof. It is a straightforward verification of Silverman-Toeplitz Theorem conditions. □

Proposition 6.2. *Let* $A = \{\ a_{ij}\ \}$*, i = 1, 2, … , j = 0, 1, 2, … , be a double sequence of complex scalars satisfying* $A \in B_c$*. If* $P^R_{A;h}(a) \in B_c$*, for some* $a \neq 0$*, then* $P^R_{A;h}(z) \in B_c$*, for all* z *satisfying* $|z| < |a|$*.*

Proof. From the above argument, we have $P^R_A(0) \in B_c$. We only need to prove $P^R_A(z) \in B_c$, for all $|z| < |a|$ and $z \neq 0$. From the hypothesis $P^R_A(a) \in B_c$, we have

(1) $\sup\limits_{1\leq i<\infty}\sum\limits_{j=0}^{\infty}|a_{ij}a^j| = M < \infty$;
(2) $\lim\limits_{i\to\infty} a_{ij}a^j$ exists for *j* = 1, 2, … ;
(3) $\lim\limits_{i\to\infty}\sum\limits_{j=0}^{\infty} a_{ij}a^j$ exists.

We have to show

(1) $\sup\limits_{1\leq i<\infty}\sum\limits_{j=0}^{\infty}|a_{ij}z^j| < \infty$;
(2) $\lim\limits_{i\to\infty} a_{ij}z^j$ exists for *j* = 1, 2, … ;
(3) $\lim\limits_{i\to\infty}\sum\limits_{j=0}^{\infty} a_{ij}z^j$ exists.

In fact, from condition (1) for $P_A^R(a)$, we obtain

$$\sup_{1\leq i<\infty}\sum_{j=0}^{\infty}|a_{ij}z^j| = \sup_{1\leq i<\infty}\sum_{j=0}^{\infty}|a_{ij}a^j| \ \left|\frac{z}{a}\right|^j \leq \sup_{1\leq i<\infty}\sum_{j=0}^{\infty}|a_{ij}a^j| = M < \infty.$$

So $P_A^R(z)$ satisfies its condition (*a*). Similarly, from the condition (2) of $P_A^R(a)$, we have

$$\lim_{i\to\infty} a_{ij}z^j = \lim_{i\to\infty} a_{ij}a^j\left(\frac{z}{a}\right)^j = 0, \text{for} j = 1, 2, \ldots.$$

So $P_A^R(z)$ satisfies its condition (*b*). Next we show that $P_A^R(z)$ satisfies its condition (*c*) from Lemma 2.2

For any given $\varepsilon > 0$, there exists *N*, such that $|\frac{z}{a}|^{N-1} < \frac{\varepsilon}{4M}$. From conditions (1) and (2) above, there exists $K > 0$ such that for all *m*, $n > K$, the following inequality holds

$$|a_{mj}a^j - a_{nj}a^j| < \frac{\varepsilon}{2N}.$$

Now for all $m, n > K$, we have

$$\begin{aligned}
&\left|\sum_{j=0}^{\infty}a_{mj}z^j - \sum_{j=0}^{\infty}a_{nj}z^j\right| \leqq \\
\leqq\ & \left|\sum_{j=0}^{N-1}a_{mj}z^j - \sum_{j=0}^{N-1}a_{nj}z^j\right| + \left|\sum_{j=N}^{\infty}a_{mj}z^j - \sum_{j=N}^{\infty}a_{nj}z^j\right| \\
\leqq\ & \sum_{j=0}^{N-1}|a_{mj}a^j - a_{nj}a^j|\left|\frac{z}{a}\right|^j + \\
& \left|\frac{z}{a}\right|^{N-1}\left(\sum_{j=N}^{\infty}|a_{mj}a^j|\left|\frac{z}{a}\right|^{j-N+1} + \sum_{j=N}^{\infty}|a_{nj}a^j|\left|\frac{z}{a}\right|^{j-N+1}\right) \\
\leqq\ & \sum_{j=0}^{N-1}|a_{mj}a^j - a_{nj}a^j| + \left|\frac{z}{a}\right|^{N-1}\left(\sum_{j=N}^{\infty}|a_{mj}a^j| + \sum_{j=N}^{\infty}|a_{nj}a^j|\right) \\
<\ & \frac{N\varepsilon}{2N} + \frac{\varepsilon}{4M}(M+M) \\
=\ & \varepsilon.
\end{aligned}$$

This proposition is proved. ☐

Proposition 6.2 indicates that, the row power matrix $P_A^R(z)$ has a similar property to power series: If there exists a number $a \neq 0$, such that $P_A^R(a) \in B_c$, then there exists a positive number r_A such that, $P_A^R(z) \in B_c$, for all $|z| < r_A$, and $P_A^R(z) \notin B_c$, for all $|z| > r_A$. r_A is called the *radius of summability of the matrix A*. The radius of summability of the matrix A is 0, if there does not exist a number $a \neq 0$, such that $P_A^R(a) \in B_c$; The radius of summability of the matrix A is ∞, if $P_A^R(a) \in B_c$ for all numbers *a*.

We will show below that this radius of summability exists also for double sequences of second and third type.

The following corollary follows immediately from Proposition 6.2 and the above notations.

Corollary 6.3. *3. Let A = $\{ a_{ij} \}$, i = 1, 2, … , j = 0, 1, 2, … , be a double sequence of complex scalars. If $A \in B_c$ then $r_A \geq 1$.*

For any given row power matrix $P_A^R(z)$, the entries of any fixed row, *i*, can be treated as the terms of a power series

$$\sum_{j=0}^{\infty} a_{ij} z^j$$

Its radius of convergence is denoted by r_A^i , for *i* = 1, 2, … .

Proposition 6.4. $r_A \leqq \underset{1\leq i<\infty}{\mathit{inf}}\, r_A^i$.

Proof. For any given $|z| < r_A$, $P_A^R(z) \in B_c$, we have $\lim_{i\to\infty} \sum_{j=0}^{\infty} a_{ij} z^j$ exists. The series $\sum_{j=0}^{\infty} a_{ij} z^j$ is convergent, for $i = 1, 2, \ldots$. It implies that $|z| \leqq r_A^i$, for *i* = 1, 2, … . It completes the proof of this proposition. ☐

For power double sequences of second type we have:

Proposition 6.5. *Let $\{a_{ij}\}$, $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$, be a double sequence of complex scalars and g a complex power series.*

If $P_{A;g}(a) \in B_c$, for some $a \neq 0$, then $P_{A;g}(z) \in B_c$, for all z satisfying $|z| < |a|$.

Proof. It is clear that $P_{A;g}(0) \in B_c$ and we only need to prove $P_{A;g}(z) \in B_c$, for all $0 < |z| < |a|$. From the hypothesis $P_{A;g}(a) \in B_c$, we have

(a) $\sup_{0\leq i<\infty} \sum_{j=0}^{\infty} |a_{ij} g_{i+j} a^{i+j}| = M < \infty$;
(b) $\lim_{i\to\infty} a_{ij} g_{i+j} a^{i+j}$ exists for *j* = 1, 2, … ;
(c) $\lim_{i\to\infty} \sum_{j=0}^{\infty} a_{ij} g_{i+j} a^{i+j}$ exists.

We have to show that the above three conditions are also true for $P_{A;g}(z)$:

(a) $\sup_{0\leq i<\infty} \sum_{j=0}^{\infty} |a_{ij} g_{i+j} z^{i+j}| < \infty$;
(b) $\lim_{i\to\infty} a_{ij} g_{i+j} z^{i+j}$ exists for *j* = 0,1,2, … ;

(c) $\lim_{i\to\infty}\sum_{j=0}^{\infty} a_{ij}g_{i+j}z^{i+j}$ exists.

In fact, from the condition (*a*) for $P_{A;g}(a)$, we obtain

$$\begin{aligned}\sup_{0\le i<\infty}\sum_{j=0}^{\infty}|a_{ij}g_{i+j}z^{i+j}| &= \sup_{0\le i<\infty}\sum_{j=0}^{\infty}|a_{ij}g_{i+j}a^{i+j}|\left|\frac{z}{a}\right|^{i+j}\\ &\le \sup_{0\le i<\infty}\sum_{j=0}^{\infty}|a_{ij}g_{i+j}a^{i+j}| = M<\infty\end{aligned}$$

So $P_{A;g}(z)$ satisfies its condition (*a*). Similarly, from the condition (*b*) of $P_{A;g}(a)$, we have

$$\lim_{i\to\infty} a_{ij}g_{i+j}z^{i+j} = \lim_{i\to\infty} a_{ij}g_{i+j}a^{i+j}\left(\frac{z}{a}\right)^{i+j} = 0, \text{for} j=0,1,2,\ldots$$

So $P_{A;g}(z)$ satisfies its condition (*b*). Next we show that $P_{A;g}(z)$ satisfies its condition (*c*).
For any given $\varepsilon>0$, there exists *N*, such that $|\frac{z}{a}|^{N-1}<\frac{\varepsilon}{4M}$. From conditions (*a*) and (*b*) above, where we set $k=i+j$, there exists $K>0$ such that $|\frac{z}{a}|^{K}<\frac{\varepsilon}{4MN}$ and for all *m*, $n>K$, the following inequalities hold (without loss of generality $m<n$)

$$\sum_{j=0}^{\infty}|a_{m-j,j}g_m a^m|<M, \sum_{j=0}^{\infty}|a_{n-j,j}g_n a^n|<M,$$

$$|a_{m-j,j}g_m a^m - a_{n-j,j}g_n a^n| < \frac{\varepsilon}{4N}.$$

Now for all *m*, $n>K$, we have

$$\begin{aligned}&\left|\sum_{j=0}^{\infty}a_{m-j,j}g_m z^m - \sum_{j=0}^{\infty}a_{n-j,j}g_n z^n\right| \leqq\\
\leqq\ & \left|\sum_{j=0}^{N-1}a_{m-j,j}g_m z^m - \sum_{j=0}^{N-1}a_{n-j,j}g_n z^n\right| + \left|\sum_{j=N}^{\infty}a_{m-j,j}g_m z^m - \sum_{j=N}^{\infty}a_{n-j,j}g_n z^n\right|\\
\leqq\ & \sum_{j=0}^{N-1}\left|a_{m-j,j}g_m a^m\left|\frac{z}{a}\right|^m - a_{n-j,j}g_n a^n\left|\frac{z}{a}\right|^n\right| +\\
& \left|\frac{z}{a}\right|^{N-1}\left(\sum_{j=N}^{\infty}|a_{m-j,j}g_m a^m|\left|\frac{z}{a}\right|^{m-N+1} + \sum_{j=N}^{\infty}|a_{n-j,j}g_n a^n|\left|\frac{z}{a}\right|^{n-N+1}\right)\\
\leqq\ & \sum_{j=0}^{N-1}\left(|a_{m-j,j}g_m a^m - a_{n-j,j}g_n a^n|\cdot\left|\frac{z}{a}\right|^m + |a_{n-j,j}g_n a^n|\cdot(1-\left|\frac{z}{a}\right|^{n-m})\cdot\left|\frac{z}{a}\right|^m\right) +\\
& \left|\frac{z}{a}\right|^{N-1}\left(\sum_{j=N}^{\infty}|a_{m-j,j}g_m a^m| + \sum_{j=N}^{\infty}|a_{n-j,j}g_n a^n|\right)\\
<\ & \frac{N\varepsilon}{4N} + NM\frac{\varepsilon}{4MN} + \frac{\varepsilon}{4M}(M+M)\\
=\ & \varepsilon.\end{aligned}$$

This proposition is proved. □

For power double sequences of third type we have:

Proposition 6.6. *Let $\{a_{ij}\}$, $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots$, be a double sequence of complex scalars and g, h complex power series. If $P_{A;g,h}(a) \in B_c$, for some $a \neq 0$, then $P_{A;g,h}(z) \in B_c$, for all z satisfying $|z| < |a|$.*

Proof. This proof is very similar to the one of the Proposition 6.5 above and will be omitted. □

Remark 6.7. Propositions 6.5 and 6.6 might seem to generalize the Proposition 6.1 or Proposition 6.2 but they don't. We cannot write $P_A^R(z)$ nor $P_A^C(z)$ in the form of $P_{A;g}$ or $P_{A;g,h}$. But we can do so for $P_A(z)$ by choosing $g_i = 1$ or $g_i = h_i = 1$ respectively.

Obviously $P_{A;g}(0) \in B_c$, $A = P_{A;g}(1) \in B_c$, $P_{A;g,h}(0) \in B_c$, and $A = P_{A;g,h}(1) \in B_c$. Then for these power matrices $r_A \geq 1$.

7. Applications to Analytic Functions and Examples

Proposition 7.1. *Let $f(z)$ be an analytic function with power series expansion $\sum_{k=0}^{\infty} p_k z^k$ about point 0 with the radius of convergence $r > 1$. Then, for any $s = [s_k] = [s_1,\ s_2,\ s_3,\ \ldots] \in c$ and for any $|z| < r$, the sequence $[\sum_{k=0}^{n} p_k s_{k+1} z^k]$, n = 1, 2, … , is also convergent.*

Proof. Define $A = \{\ a_{ij}\ \}$, i = 1, 2, 3, … , j = 0, 1, 2, … , as follows:

$$a_{ij} = \begin{cases} p_j, j < i \\ 0, j \geq i \end{cases}$$

Then A and $A(z)$ have the following properties:

(a) $A \in B_c$, with $M = \sum_{k=0}^{\infty} p_k$;
(b) $r_A^i = \infty$, for i = 1, 2, …;
(c) $r_A = r$;
(d) $A(z) \in B_c$.

And the proposition follows from the property (d). □

For the following examples, one can check the conditions listed.

Example 7.2. Let $A = \{i + j\}$, i = 1, 2, …, j = 0, 1, 2, … :

(a) $A \notin B_c$;
(b) $r_A^i = 1$, for i = 1, 2, … ;
(c) $r_A = 1$.

Example 7.3. Let $A = \{\ \frac{1}{i+j}\ \}$, i = 1, 2, … , j = 0, 1, 2, … :

(a) $A \notin B_c$;
(b) $r_A^i = 1$, for i = 1, 2, … ;
(c) $r_A = 1$.

Example 7.4. Let $A = \{\ (i + j)!\ \}$, i = 1, 2, … , j = 0, 1, 2, … :

(a) $A \notin B_c$;
(b) $r_A^i = 0$, for i = 1, 2, … ;
(c) $r_A = 0$.

Example 7.5. Let $A = \{\ \frac{1}{(i+j)!}\ \}$, i = 1, 2, … , j = 0, 1, 2, … :

(a) $A \in B_c$, with $M = \sum\limits_{j=0}^{\infty} \frac{1}{j!}$;
(b) $r_A^i = \infty$, for i = 1, 2, …;
(c) $r_A = \infty$.

Example 7.6. For a given $b > 0$, let A = { b^{i+j} }, i = 1, 2, … , j = 0, 1, 2, … :
(a) $A \in B_c$, if b ¡ 1, with $M = \frac{1}{1-b}$, and $A \notin B_c$, if $b \geq 1$;
(b) $r_A^i = b^{-1}$, for i = 1, 2, … ;
(c) $r_A = b^{-1}$.

Example 7.7. Let $[p_k]$ be a sequence of positive numbers satisfying $\sum\limits_{k=1}^{\infty} p_k < \infty$, and let $P_i = \sum\limits_{k=1}^{i} p_k$. Define $A = a_{ij},\ i = 1, 2, ...,\ j = 0, 1, 2, ...$, as follows

$$a_{ij} = \begin{cases} \frac{1}{P_i}, j < i \\ 0, j \geq i \end{cases}:$$

(a) $A \in B_c$, with $M = \frac{1}{p_1}$;
(b) $r_A^i = \infty$, for i = 1, 2, …;
(c) $r_A = 1$;
(d) A is not a regular method of summability.

Example 7.8. Let $[p_k]$ be a sequence of positive numbers satisfying $\sum\limits_{k=1}^{\infty} p_k = \infty$, and let $P_i = \sum\limits_{k=1}^{i} p_k$. Define $A = a_{ij}$, as in Example 7.7 Then:
(a) $A \in B_c$, with $M = \frac{1}{p_1}$;
(b) $r_A^i = \infty$, for i = 1, 2, …;
(c) $r_A = \sup\{b > 0 : \lim\limits_{i \to \infty} \frac{b^i}{P_i} \text{ exists}\} < \infty$;
(d) A is a regular method of summability.

8. Conclusion

In this paper, we introduced the concept of parametric summability for parameterized double sequences. We proved some results of parametric summability and radius of summability of some parameterized double sequences. We also gave some applications of parametric summability of parameterized to standard summability theory and analytic functions. Several examples were provided to demonstrate the main results of this paper.

We have some ideas to extend the parametric summability for the further study.

1. In section 4 of this paper, we studied the radius of summability of some parameterized double sequences. By the same ideas, we may consider the "convergent speed" of a given parameterized double sequence.

2. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be a real Banach spaces. Suppose that both X and Y have Schauder bases $\{e_n\}$ and $\{d_n\}$, respectively. Let $T : X \to Y$ be a linear and continuous (bounded) mapping. Then, there is a real infinite matrix $A = (a_{i,j})$, with $i, j = 0, 1, 2, ...$ a double sequence of real constants, such that, for any $x = \sum_{n=0}^{\infty} a_n e_n$, we have

$$\begin{aligned} Tx &= \textstyle\sum_{n=0}^{\infty} a_n T e_n \\ &= \textstyle\sum_{n=0}^{\infty} a_n (\sum_{j=0}^{\infty} a_{n,j} d_j) \\ &= \textstyle\sum_{j=0}^{\infty} (\sum_{n=0}^{\infty} a_n a_{n,j}) d_j. \end{aligned}$$

Then, by the fact that the linear and continuous (bounded) mapping $T : X \to Y$ is induced by the real infinite matrix $A = (a_{i,j})$, we may study the parameterized linear and continuous (bounded) mappings from X and Y:

(i) $T^C : X \to Y$ that is induced by the column power matrix $(a_{i,j}z^i)$
(ii) $T^R : X \to Y$ that is induced by the row power matrix $(a_{i,j}z^j)$

3. What is the connection between the norms of T^C, T^R and the norm of T?

4. What is the radius r^C of the linear and continuous mapping $T^C : X \to Y$? More precisely, we want to find r^C such that, if $|z| < r^C$, then for any $x = \sum_{n=0}^{\infty} a_n e_n$, we have $T^C x \in Y$ such that

$$\begin{aligned} T^C x &= \textstyle\sum_{n=0}^{\infty} a_n T^C e_n \\ &= \textstyle\sum_{n=0}^{\infty} a_n (\sum_{j=0}^{\infty} a_{n,j} z^n d_j) \\ &= \textstyle\sum_{j=0}^{\infty} (\sum_{n=0}^{\infty} a_n a_{n,j} z^n) d_j. \end{aligned}$$

5. What is the radius r^R of the linear and continuous mapping $T^C : X \to Y$? More precisely, we want to find r^R such that, if $|z| < r^R$, then for any $x = \sum_{n=0}^{\infty} a_n e_n$, we have $T^R x \in Y$ such that

$$\begin{aligned} T^R x &= \textstyle\sum_{n=0}^{\infty} a_n T^R e_n \\ &= \textstyle\sum_{n=0}^{\infty} a_n (\sum_{j=0}^{\infty} a_{n,j} z^j d_j) \\ &= \textstyle\sum_{j=0}^{\infty} (\sum_{n=0}^{\infty} a_n a_{n,j} z^j) d_j. \end{aligned}$$

6. What is the connection between the radius r^C and the radius r^R?

7. Let $(H, \|\cdot\|)$ and $(K, \|\cdot\|)$ be real Hilbert spaces. Suppose that both H and K have orthonormal bases. One may consider here the same questions as above from number 2 to number 6 about linear and continuous (bounded) mappings from H to K.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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