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ON NONSMOOTH MULTIOBJECTIVE SEMI-INFINITE MATHEMATICAL PROGRAMMING PROBLEMS WITH VANISHING CONSTRAINTS ON HADAMARD MANIFOLDS: SADDLE POINT OPTIMALITY AND LAGRANGE DUALITY

ARNAV GHOSH[1,*] AND JEN-CHIH YAO[1,2]

[1] *Center for General Education, China Medical University, Taichung, Taiwan*
[2] *Academy of Romanian Scientists, 50044 Bucharest, Romania*

Abstract. In this article, we investigate a class of nonsmooth multiobjective semi-infinite programming problems with vanishing constraints (in short, NSIMOPVC) on Hadamard manifolds. We introduce the scalarized Lagrange-type dual problem and the vector Lagrange-type dual problem related to NSIMOPVC in the framework of Hadamard manifolds. We derive weak and strong duality theorems relating NSIMOPVC and the corresponding dual problems under suitable geodesic convexity assumptions. Moreover, we establish scalarized saddle point optimality conditions as well as vector saddle point optimality conditions for NSIMOPVC in the setting of Hadamard manifolds. To the best of our knowledge, this is for the first time that saddle point optimality criteria and Lagrange duality for NSIMOPVC have been investigated on Hadamard manifolds.

Keywords. Multiobjective optimization, Hadamard manfiolds, Saddle point optimality, Lagrange duality.

1. Introduction

In optimization theory, a semi-infinite programming problem (in short, SIP) refers to the class of mathematical programming problems in finitely many decision variables on some feasible set described by infinitely many constraints. The origin of SIP is closely related to the work of Haar [10], Chebyshev approximations and Fritz John optimality criteria [15]. Later, the term '(SIP)' was coined by Charnes et al. [3] in 1962. (SIP) has a very wide range of applications in several real-life problems of mathematical physics, game theory, engineering design, etc., see [6, 11]. As a result, in recent times, SIP has emerged as a very crucial area of research in optimization and has been studied extensively by numerous researchers, see, for instance [4, 8, 19, 25] and the references cited therein.

Mathematical programming problems with vanishing constraints (abbreviated as, MPVC) comprise a special category of constrained optimization problems. One of the important features of MPVCs lies in the fact that in various applications, some of the constraints present often become redundant at certain feasible elements. MPVCs were introduced by Achtziger and Kanzow [1]. In the last few years, various practical problems in the field of structural and topology optimization have been modeled as MPVC. For a more comprehensive study of (MPVCs), we refer to [12, 13, 14, 24].

Hoheisel and Kanzow [12] derived first-order and second-order optimality conditions for MPVC. Further, constraint qualifications for MPVC were investigated by Hoheisel and Kanzow [14]. Dorsch et al. [5] investigated several stationary points of MPVC. Guu et al. [9] derived sufficient optimality criteria for semi-infinite MPVC. Mishra et al. [20] explored constraint qualifications for MPVC with vector-valued objective function. Upadhyay and Ghosh [23] developed constraint qualifications for

*Corresponding author.
E-mail address: arnavghosh21@gmail.com(A. Ghosh), yaojc@mail.cmu.edu.tw (J.-C. Yao)
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MPVC on Hadamard manifolds. Upadhyay et al. [24] established Karush-Kuhn-Tucker type optimality conditions and duality for nonsmooth multiobjective semi-infinite MPVC on Hadamard manifolds.

In the last few decades, the study of optimization problems in the framework of manifolds has emerged as a very prominent and interesting area of research. In the modeling and analysis of many practical problems in the area of data analysis, the framework of manifolds is found to be more useful, as compared to the traditional Euclidean space setting (for reference, see, [7]). On the other hand, generalizing various methods of optimization from Euclidean spaces to the framework of manifolds has numerous important advantages. For instance, it is possible to remodel many difficult, constrained, non-convex optimization problems formulated in the Euclidean space framework to simpler equivalent unconstrained convex problems in the Hadamard manifold framework, see [22]. In light of these advantages, several researchers have generalized many interesting ideas related to optimization theory from Euclidean space setting to the framework manifolds, see, [24] and the references provided therein.

In sharp contrast to Euclidean spaces, manifolds, in general, are not equipped with a linear structure, though globally diffeomorphic. Therefore, despite being globally homeomorphic to Euclidean spaces, the development of optimization techniques in the framework of Hadamard manifolds is accompanied by several difficulties. For instance, in sharp contrast to the setting of Euclidean space, the notion of a unique line segment joining any two points is not available in the manifold setting. Further, the exponential map and inverse of the exponential map on Hadamard manifolds are nonlinear (see, for instance, [18]). As a result, new techniques have been developed by researchers in the last few decades to investigate optimization problems on manifolds. For instance, the concept of geodesic convexity is introduced in the manifold setting, employing the notion of unique minimal geodesic to connect any two points in the Hadamard manifold. Moreover, the concepts of parallel transport and exponential maps on the tangent space of a Hadamard manifold (which has a vector space structure) are employed in order to deal with the nonlinearity of manifolds. The primary motivation and objective to investigate nonsmooth MPVC on Hadamard manifolds, rather than Riemannian manifolds, is as follows. Firstly, the notion of injectivity radius (see, for instance, [26]) ensures that the exponential map is globally diffeomorphic in the case of Hadamard manifolds. On the other hand, the exponential map is locally diffeomorphic in the setting of Riemannian manifolds. Therefore, the established results in the framework of Hadamard manifolds hold within the totally normal neighborhood of each point in the Riemannian manifolds.

Motivated by the works of [21, 24], in this article, we study a class of nonsmooth multiobjective semi-infinite programming problems with vanishing constraints (in short, NSIMOPVC) on Hadamard manifolds. We introduce the scalarized Lagrange-type dual problem and the vector Lagrange-type dual problem related to NSIMOPVC, in the framework of Hadamard manifolds. We derive weak and strong duality theorems relating NSIMOPVC and the corresponding dual problems under suitable geodesic convexity assumptions. Moreover, we establish saddle point optimality conditions for NSIMOPVC in the setting of Hadamard manifolds. The results derived in this paper generalize, extend and unify several notable results existing in the literature. For instance, the results derived in the paper extend the corresponding results derived by Tung et al. [21] from Euclidean space setting to the framework of Hadamard manifold, as well as generalize them for a more general class of optimization problems.

The rest of the article unfolds in the following manner. In Section 2, we recall some basic definitions and notation that will be used throughout the paper. In Section 3, we formulate scalarized Lagrange type dual problem related to NSIMOPVC and establish several duality results. We also discuss scalar saddle point optimality criteria for NSIMOPVC. In Section 4, we formulate the vector Lagrange type dual problem related to NSIMOPVC and establish weak and strong duality results. Moreover, vector saddle point optimality criteria for NSIMOPVC are established. In Section 5, conclusions are drawn and some future directions are discussed.

2. Notation and Mathematical Preliminaries

The standard symbols $\mathbb{N}$ and $\mathbb{R}^n$ are used to signify the set consisting of every natural number and the Euclidean space having dimension n ($n \in \mathbb{N}$), respectively. The symbol $\emptyset$ is used to signify an empty set. The non-negative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$. Let $\mathcal{A}$ be any arbitrary infinite set. Then the following sets will be used in the sequel:

$$\mathbb{R}^{|\mathcal{A}|} := \{(x_\ell)_{\ell \in \mathcal{A}} : x_\ell = 0 \text{ for every } \ell \in \mathcal{A}, \text{ except } x_\ell \neq 0 \text{ for finitely many } \ell \in \mathcal{A}\},$$
$$\mathbb{R}^{|\mathcal{A}|}_+ := \{x = (x_\ell)_{\ell \in \mathcal{A}} \in \mathbb{R}^{|\mathcal{A}|} : x_\ell \geq 0, \ \forall \ell \in \mathcal{A}\}.$$

Let $c, d \in \mathbb{R}^n$. The following notations will be used in the paper:

$$c \prec d \iff c_j < d_j, \quad \forall j = 1, 2, \ldots, n.$$
$$c \preceq d \iff \begin{cases} c_j \leq d_j, & \forall j = 1, 2, \ldots, n, \\ c_r < d_r, & \text{for at least one } r \in \{1, 2, \ldots, n\}. \end{cases}$$

Let $\mathcal{B} \subset \mathbb{R}^n$. The linear hull, closure and convex hull of $\mathcal{B}$ in $\mathbb{R}^n$ are denoted by the symbols $\operatorname{span}(\mathcal{B})$, $\operatorname{cl}(\mathcal{B})$ and $\operatorname{co}(\mathcal{B})$, respectively. The positive conic hull of $\mathcal{B}$ is signified by the symbol $\operatorname{pos}(\mathcal{B})$. The following sets will be employed in the sequel:

$$\mathcal{B}^- := \{u \in \mathbb{R}^n : u^T v \leq 0, \ \forall v \in \mathcal{B}\},$$
$$\mathcal{B}^s := \{u \in \mathbb{R}^n : u^T v < 0, \ \forall v \in \mathcal{B}\},$$
$$\mathcal{B}^\perp := \{u \in \mathbb{R}^n : u^T v = 0, \ \forall v \in \mathcal{B}\}.$$

Let $\mathcal{B}_1, \mathcal{B}_2 \subset \mathbb{R}^n$. The following relations are well-known:

$$\operatorname{pos}(\mathcal{B}_1 \cup \mathcal{B}_2) = \operatorname{pos}(\mathcal{B}_1) + \operatorname{pos}(\mathcal{B}_2), \quad \operatorname{span}(\mathcal{B}_1 \cup \mathcal{B}_2) = \operatorname{span}(\mathcal{B}_1) + \operatorname{span}(\mathcal{B}_2).$$

The following notation will be employed in the sequel:

$$\underset{\mathbb{R}^m_+}{\operatorname{WMin}} \ S = \left\{ \bar{s} \in S \mid (S - \bar{s}) \cap - \operatorname{int} \mathbb{R}^m_+ = \emptyset \right\}.$$

The notation $\mathscr{M}$ will be used to signify a smooth manifold having dimension n, where $n \in \mathbb{N}$. Let $y^* \in \mathscr{M}$. The set of all tangent vectors at y^* is known as the tangent space at y^*, and is denoted by $T_{y^*}\mathscr{M}$. Further, $T_{y^*}\mathscr{M}$ is a real linear space, having a dimension n. In case we are restricted to real manifolds, $T_{y^*}\mathscr{M}$ is isomorphic to $\mathbb{R}^n$. A Riemannian metric $\mathscr{G}$ on $\mathscr{M}$ is a 2-tensor field that is symmetric as well as positive-definite. For any $w_1, w_2 \in T_{y^*}\mathscr{M}$, the inner product of w_1 and w_2 is given by $\langle w_1, w_2 \rangle_{y^*} = \mathscr{G}_{y^*}(w_1, w_2)$, where $\mathscr{G}_{y^*}$ denotes the Riemannian metric at the element $y^* \in \mathscr{M}$. The norm corresponding to the inner product $\langle w_1, w_2 \rangle_{y^*}$ is denoted by $\| \cdot \|_{y^*}$ (or simply, $\| \cdot \|$, when there is no ambiguity regarding the subscript). Let $a, b \in \mathbb{R}$, $a < b$ and $\nu : [a, b] \to \mathscr{M}$ be any piecewise differentiable curve, such that $\nu(a) = y^*$, $\nu(b) = \hat{z}$. A vector field Y is referred to be parallel along the curve ν, provided that the $\nabla_{\nu'} Y = 0$. If $\nabla_{\nu'} \nu' = 0$, then ν is termed as a geodesic. If $\|\nu\| = 1$, then ν is said to be normalised. The exponential function $\exp_{y^*} : T_{y^*}\mathscr{M} \to \mathscr{M}$ is given by $\exp_{y^*}(\hat{w}) = \nu(1)$, where ν is a geodesic which satisfies $\nu(0) = y^*$ and $\nu'(0) = \hat{w}$. A Riemannian manifold is referred to as a Hadamard manifold (or, Cartan-Hadamard manifold) provided that $\mathscr{M}$ is connected, geodesic complete, and has a nonpositive sectional curvature throughout. Henceforth, in our discussions, the notation $\mathscr{M}$ will always signify a Hadamard manifold of dimension n, unless it is specified otherwise.

Let $y^* \in \mathscr{M}$. Then, $\exp_{y^*} : T_{y^*}\mathscr{M} \to \mathscr{M}$ is a globally diffeomorphic function. Moreover, $\exp^{-1}_{y^*} : \mathscr{M} \to T_{y^*}\mathscr{M}$ satisfies $\exp^{-1}_{y^*}(y^*) = 0$. Furthermore, for any $y_1^*, y_2^* \in \mathscr{M}$, there always exists some unique normalized minimal geodesic $\nu_{y_1^*, y_2^*} : [0, 1] \to \mathscr{M}$, such that $\nu_{y_1^*, y_2^*}(\tau) = \exp_{y_1^*}(\tau \exp^{-1}_{y_1^*}(y_2^*))$, for every $\tau \in [0, 1]$. Thus, every Hadamard manifold $\mathscr{M}$ of dimension n is diffeomorphic to $\mathbb{R}^n$.

The following definition of contingent cone is from [16].

Definition 2.1. Let $\mathcal{F} \subseteq \mathscr{M}$ and $z \in \mathrm{cl}(\mathcal{F})$. Then the contingent cone (in other terms, Bouligand tangent cone) of $\mathcal{F}$ at z, denoted by $\mathscr{T}(\mathcal{F}, z)$, is defined as follows:

$$\mathscr{T}(\mathcal{F}, z) := \{w \in T_z\mathscr{M} : \exists t_n \downarrow 0, \exists w_n \in T_z\mathscr{M}, w_n \to w, \exp_z(t_n w_n) \in \mathcal{F}, \forall n \in \mathbb{N}\}.$$

The following definition is from Udriste [22].

Definition 2.2. Any set $\mathcal{D} \subseteq \mathscr{M}$ is termed as geodesic convex set, provided that for every pair of distinct elements $z_1, z_2 \in \mathcal{D}$ and for the geodesic $\gamma_{z_1,z_2} : [0,1] \to \mathscr{M}$ connecting z_1 and z_2, we have $\gamma_{z_1,z_2}(t) \in \mathcal{D}$, for every $t \in [0,1]$, where, $\gamma_{z_1,z_2}(t) = \exp_{z_1}\left(t \exp_{z_1}^{-1}(z_2)\right)$.

The following definitions and theorem will be employed in the sequel (see, [2]).

Definition 2.3. Let $\mathcal{A} \subseteq \mathscr{M}$ and $\varphi : \mathcal{A} \to \mathbb{R}$ be a real-valued function. Then φ is said to be locally Lipschitz at $z \in \mathcal{A}$ with rank $\mathcal{L}$ ($\mathcal{L} \in \mathbb{R}$, $\mathcal{L} > 0$), if for every z_1, z_2 in some open neighborhood of z, the following is satisfied:

$$|\varphi(z_1) - \varphi(z_2)| \leq \mathcal{L}\, \omega(z_1, z_2),$$

where $\omega(z_1, z_2)$ is the Riemannian distance between the points z_1 and z_2 on $\mathcal{A}$.

Definition 2.4. Let $z_1, z_2 \in \mathscr{M}$ and $\varphi : \mathscr{M} \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of φ at z_2 in the direction $w \in T_{z_2}\mathscr{M}$, denoted by the symbol $\varphi^\circ(z_2; w)$, is defined as follows:

$$\varphi^\circ(z_2; w) := \limsup_{z_1 \to z_2,\, t \downarrow 0} \frac{\varphi\left(\exp_{z_1}\left(t \left(d\exp_{z_2}\right)_{\exp_{z_2}^{-1}(z_1)} w\right)\right) - \varphi(z_1)}{t},$$

where $\left(d\exp_{z_2}\right)_{\exp_{z_2}^{-1}(z_1)} : T_{\exp_{z_2}^{-1}(z_1)}\left(T_{z_2}\mathscr{M}\right) \simeq T_{z_2}\mathscr{M} \to T_{z_1}\mathscr{M}$ is the differential of the exponential function at $\exp_{z_2}^{-1}(z_1)$.

Definition 2.5. Let $\varphi : \mathscr{M} \to \mathbb{R}$ be a locally Lipschitz function. The Clarke subdifferential of φ at $z_1 \in \mathscr{M}$, denoted by $\partial^c\varphi(z_1)$, is defined as follows:

$$\partial^c\varphi(z_1) := \left\{\xi \in T_{z_1}\mathscr{M} : \varphi^\circ(z_1; w) \geq \langle \xi, w\rangle_{z_1},\ \forall w \in T_{z_1}\mathscr{M}\right\}.$$

Lemma 2.6. *Let $\varphi : \mathscr{M} \to \mathbb{R}$ be a locally Lipschitz function with Lipschitz constant $\mathcal{L}$ near $y \in \mathscr{M}$. Then*

(a) The Clarke subdifferential $\partial^c\varphi(y)$ is a nonempty, convex, compact subset of $T_y\mathscr{M}$. Moreover, $\|\xi\| \leq \mathcal{L}$ for every *$\xi \in \partial^c\varphi(y)$, and $\partial^c\varphi(y)$ is upper semicontinuous at y.*

(b) For every w in $T_y\mathscr{M}$, we have: $\varphi^\circ(y; w) = \max\{\langle \xi, w\rangle : \xi \in \partial^c\varphi(y)\}$.

(c) If $\{y_n\}$ and $\{\xi_n\}$ are sequences in $\mathscr{M}$ and the tangent bundle $T\mathscr{M}$, respectively, such that $\xi_n \in \partial^c\varphi(y_n)$ for every $n \in \mathbb{N}$, $\{y_n\} \to y$. Further, let ξ be a cluster point of $\{\xi_n\}$. Then, we have $\xi \in \partial^c\varphi(y)$.

Theorem 2.7. *Let $\varphi : \mathscr{M} \to \mathbb{R}$ be a locally Lipschitz function. Then, for any $z_1, z_2 \in \mathscr{M}$, there exist some $t_0 \in (0,1)$ and $z_0 = \sigma(t_0)$, such that $\varphi(z_2) - \varphi(z_1) \in \langle \partial^c\varphi(z_0), \sigma'(t_0)\rangle_{z_0}$, where $\sigma(t) :=$* $\exp_{z_2}\left(t \exp_{z_2}^{-1}(z_1)\right)$, $t \in [0,1]$.

Definition 2.8. Let $\varphi : \mathcal{D} \to \mathbb{R}$ be any locally Lipschitz function defined on a geodesic convex subset $\mathcal{D}$ of $\mathscr{M}$. Then, φ is termed as geodesic convex (respectively, strictly geodesic convex) at z_2, provided that for every $z_1 \in \mathcal{D}$ (respectively, $z_1 \in \mathcal{D}$, $z_1 \neq z_2$) and for every $\xi \in \partial^c\varphi(z_2)$, we have:

$$\varphi(z_1) - \varphi(z_2) \geq (\text{respectively}\ , >) \left\langle \xi, \exp_{z_2}^{-1}(z_1)\right\rangle_{z_2}.$$

3. Scalarized Lagrange Duality and Saddle Point Optimality for NSIMOPVC

We consider the following nonsmooth multiobjective semi-infinite programming problem with equilibrium constraints defined on Hadamard manifolds (NSIMOPVC):

$$\begin{aligned}
\text{(NSIMOPVC)} \quad \text{Minimize} \quad & \Phi(y) := (\Phi_1(y), \ldots, \Phi_m(y)), \\
\text{subject to} \quad & \Psi_t(y) \leq 0, \quad \forall t \in \mathcal{T}, \\
& \theta_j(y) = 0, \quad \forall j \in \mathcal{J} := \{1, \ldots, q\}, \\
& \mathcal{A}_j(y) \geq 0, \quad \forall j \in \mathcal{S} := \{1, \ldots, p\}, \\
& \mathcal{B}_j(y) \geq 0, \quad \forall j \in \mathcal{S} := \{1, \ldots, p\}, \\
& \mathcal{B}_j(y)\, \mathcal{A}_j(y) = 0, \quad \forall j \in \mathcal{S} := \{1, \ldots, p\}.
\end{aligned}$$

Here, all the functions Φ_j $(j \in I := \{1, \ldots, m\})$, Ψ_t $(t \in \mathcal{T})$, θ_j $(j \in \mathcal{J})$ and $\mathcal{B}_j$, $\mathcal{A}_j$ $(j \in \mathcal{S})$ are locally Lipschitz real-valued functions defined on some $\hat{n}$-dimensional Hadamard manifold $\mathscr{M}$ $(\hat{n} \in \mathbb{N})$. The index set $\mathcal{T}$ is considered to be arbitrary (possibly infinite).

The feasible set of the problem NSIMOPVC is denoted by $\mathcal{F}$. Let $y \in \mathcal{F}$. The following sets will be employed in the sequel:

$$L(y) := \{t \in \mathcal{T} : \Psi_t(y) = 0\},$$
$$\mathscr{A}(y) := \{\sigma \in \mathbb{R}_+^{|\mathcal{T}|} : \sigma_t \Psi_t(y) = 0, \ \forall t \in \mathcal{T}\}.$$

Now we recall the notions of Pareto efficient and weak Pareto efficient solutions for NSIMOPVC (see, for instance, [25]).

Definition 3.1. Let $\hat{a} \in \mathcal{F}$. Then the element $\hat{a}$ is termed as a Pareto efficient (respectively, weak Pareto efficient) solution of NSIMOPVC, provided that there exists no other element $a \in \mathcal{F}$, satisfying

$$\Phi(a) \preceq \text{(respectively, } \prec) \ \Phi(\hat{a}).$$

Let $\hat{a} \in \mathcal{F}$. The following index sets will be helpful in the subsequent discussions.

$$\begin{aligned}
\mathcal{R}_+(\hat{a}) &:= \{j \in \mathcal{S} : \mathcal{A}_j(\hat{a}) > 0\}, \\
\mathcal{R}_0(\hat{a}) &:= \{j \in \mathcal{S} : \mathcal{A}_j(\hat{a}) = 0\}, \\
\mathcal{R}_{+0}(\hat{a}) &:= \{j \in \mathcal{S} : \mathcal{A}_j(\hat{a}) > 0, \mathcal{B}_j(\hat{a}) = 0\}, \\
\mathcal{R}_{0+}(\hat{a}) &:= \{j \in \mathcal{S} : \mathcal{A}_j(\hat{a}) = 0, \mathcal{B}_j(\hat{a}) > 0\}, \\
\mathcal{R}_{00}(\hat{a}) &:= \{j \in \mathcal{S} : \mathcal{A}_j(\hat{a}) = 0, \mathcal{B}_j(\hat{a}) = 0\}.
\end{aligned}$$

Remark 3.2. Every index set defined above clearly depends on the choice of $\hat{a} \in \mathcal{F}$. Nevertheless, in the remaining portion of the article, we shall not indicate such dependence explicitly when it is easily perceivable from the context.

The following definition of strong stationary element of NSIMOPVC in the Hadamard manifold framework is from Upadhyay et al. [24].

Definition 3.3. Let $\hat{a} \in \mathcal{F}$. Then $\hat{a}$ is referred to as a strong stationary element of NSIMOPVC, provided that there exist $\alpha \in \mathbb{R}_+^m$, $\sigma^\Psi \in \mathscr{A}(\hat{a})$, $\sigma^\theta \in \mathbb{R}^q$, $\sigma^{\mathcal{A}} \in \mathbb{R}^p$, $\sigma^{\mathcal{B}} \in \mathbb{R}^p$, satisfying:

$$\begin{aligned}
0 \in \sum_{j \in I} \alpha_j \partial^c f_j(\hat{a}) + \sum_{t \in \mathcal{T}} \sigma_t^\Psi \partial^c \Psi_t(\hat{a}) + \sum_{j \in \mathcal{J}} \sigma_j^\theta \partial^c \theta_j(\hat{a}) \\
- \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{A}} \partial^c \mathcal{A}_j(\hat{a}) - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{B}} \partial^c \mathcal{B}_j(\hat{a}),
\end{aligned}$$

$$\sigma_j^{\mathcal{A}} = 0, \quad \forall j \in \mathcal{R}_{+0}(\hat{a}), \quad \sigma_j^{\mathcal{A}} \geq 0, \quad \forall j \in \mathcal{R}_{00}(\hat{a}),$$
$$\sigma_j^{\mathcal{B}} = 0, \quad \forall j \in \mathcal{R}_{0+}(\hat{a}),$$
$$\sigma_j^{\mathcal{B}} \geq 0, \quad \forall j \in \mathcal{R}_{00}(\hat{a}), \quad \text{and} \sum_{j \in I} \alpha_j = 1.$$

Let $\hat{a} \in \mathcal{F}$, $\sigma^\Psi \in \mathbb{R}_+^{|\mathcal{T}|}$, $\sigma^\theta \in \mathbb{R}^q$, $\sigma^{\mathcal{A}} \in \mathbb{R}^p$, $\sigma^{\mathcal{B}} \in \mathbb{R}^p$. We now define some index sets that will be employed in the rest of the article.

$$L^+(\hat{a}) := \{t \in L(\hat{a}) : \sigma_t^\Psi > 0\},$$
$$\mathcal{R}_{\mathcal{J}}^+(\hat{a}) := \{j \in \mathcal{J}(\hat{a}) : \sigma_j^\theta > 0\},$$
$$\mathcal{R}_{\mathcal{J}}^-(\hat{a}) := \{j \in \mathcal{J}(\hat{a}) : \sigma_j^\theta < 0\},$$
$$\hat{\mathcal{R}}_+^+(\hat{a}) := \{j \in \mathcal{R}_+(\hat{a}) : \sigma_j^{\mathcal{A}} > 0\},$$
$$\hat{\mathcal{R}}_0^+(\hat{a}) := \{j \in \mathcal{R}_0(\hat{a}) : \sigma_j^{\mathcal{A}} > 0\},$$
$$\hat{\mathcal{R}}_0^-(\hat{a}) := \{j \in \mathcal{R}_0(\hat{a}) : \sigma_j^{\mathcal{A}} < 0\},$$
$$\hat{\mathcal{R}}_{0+}^+(\hat{a}) := \{j \in \mathcal{R}_{0+}(\hat{a}) : \sigma_j^{\mathcal{A}} > 0\},$$
$$\hat{\mathcal{R}}_{0+}^-(\hat{a}) := \{j \in \mathcal{R}_{0+}(\hat{a}) : \sigma_j^{\mathcal{A}} < 0\},$$
$$\hat{\mathcal{R}}_{00}^+(\hat{a}) := \{j \in \mathcal{R}_{00}(\hat{a}) : \sigma_j^{\mathcal{A}} > 0\},$$
$$\hat{\mathcal{R}}_{00}^-(\hat{a}) := \{j \in \mathcal{R}_{00}(\hat{a}) : \sigma_j^{\mathcal{A}} < 0\},$$
$$\hat{\mathcal{R}}_{0-}^+(\hat{a}) := \{j \in \mathcal{R}_{0-}(\hat{a}) : \sigma_j^{\mathcal{A}} > 0\},$$
$$\mathcal{R}_{+0}^+(\hat{a}) := \{j \in \mathcal{R}_{+0}(\hat{a}) : \sigma_j^{\mathcal{B}} > 0\},$$
$$\mathcal{R}_{+0}^-(\hat{a}) := \{j \in \mathcal{R}_{+0}(\hat{a}) : \sigma_j^{\mathcal{B}} < 0\},$$
$$\mathcal{R}_{+-}^+(\hat{a}) := \{j \in \mathcal{R}_{+-}(\hat{a}) : \sigma_j^{\mathcal{B}} > 0\},$$
$$\mathcal{R}_{0+}^+(\hat{a}) := \{j \in \mathcal{R}_{0+}(\hat{a}) : \sigma_j^{\mathcal{B}} > 0\},$$
$$\mathcal{R}_{0+}^-(\hat{a}) := \{j \in \mathcal{R}_{0+}(\hat{a}) : \sigma_j^{\mathcal{B}} < 0\},$$
$$\mathcal{R}_{00}^+(\hat{a}) := \{j \in \mathcal{R}_{00}(\hat{a}) : \sigma_j^{\mathcal{B}} > 0\},$$
$$\mathcal{R}_{00}^-(\hat{a}) := \{j \in \mathcal{R}_{00}(\hat{a}) : \sigma_j^{\mathcal{B}} < 0\},$$
$$\mathcal{R}_{0-}^+(\hat{a}) := \{j \in \mathcal{R}_{0-}(\hat{a}) : \sigma_j^{\mathcal{B}} > 0\}.$$

The following definition is from [24].

Definition 3.4. Let $\hat{a} \in \mathcal{F}$. Then, the VC-linearized cone of NSIMOPVC at $\hat{a}$, denoted by $T_{\mathrm{VC}}^{\mathrm{Lin}}(\hat{a})$, is defined as follows:

$$\begin{aligned} T_{\mathrm{VC}}^{\mathrm{Lin}}(\hat{a}) := \{w \in T_{\hat{a}}\mathscr{M} : \langle \xi_t^\Psi, w\rangle_{\hat{a}} \leq 0, &\quad \forall \xi_t^\Psi \in \partial^c \Psi_t(\hat{a}),\ \forall t \in L(\hat{a}), \\ \langle \xi_j^\theta, w\rangle_{\hat{a}} = 0, &\quad \forall \xi_t^\theta \in \partial^c \theta_j(\hat{a}),\ \forall j \in \mathcal{J}, \\ \langle \xi_j^{\mathcal{M}}, w\rangle_{\hat{a}} = 0, &\quad \forall \xi_j^{\mathcal{M}} \in \partial^c \mathcal{M}_j(\hat{a}),\ \forall j \in \mathcal{R}_{0+}, \\ \langle \xi_j^{\mathcal{M}}, w\rangle_{\hat{a}} \geq 0, &\quad \forall \xi_j^{\mathcal{M}} \in \partial^c \mathcal{M}_j(\hat{a}),\ \forall j \in \mathcal{R}_{00}, \\ \langle \xi_j^{\mathcal{N}}, w\rangle_{\hat{a}} \geq 0, &\quad \forall \xi_j^{\mathcal{N}} \in \partial^c \mathcal{N}_j(\hat{a}),\ \forall j \in \mathcal{R}_{00}, \\ \langle \xi_j^{\mathcal{N}}, w\rangle_{\hat{a}} = 0, &\quad \forall \xi_j^{\mathcal{N}} \in \partial^c \mathcal{N}_j(\hat{a}),\ \forall j \in \mathcal{R}_{+0}\}. \end{aligned}$$

Now, we present NSIMOPVC-tailored ACQ in the framework of Hadamard manifolds for our considered problem.

Definition 3.5. Let $\hat{a} \in \mathcal{F}$. Then the NSIMOPVC-tailored ACQ (in short, NSIMOPVC-ACQ) holds at $\hat{a}$, if

$$T_{\mathrm{VC}}^{\mathrm{Lin}}(\hat{a}) \subseteq \mathscr{T}(\mathcal{F}, \hat{a}).$$

Let $\hat{a} \in \mathcal{F}$. We define the following sets for our convenience:

$$\mathcal{V}_\Psi := \bigcup_{t \in L} \partial^c \Psi_t(\hat{a}), \quad \mathcal{V}_\theta := \bigcup_{j \in \mathcal{J}} \partial^c \theta_j(\hat{a}), \quad \mathcal{V}_{\mathcal{A}_1} := \bigcup_{j \in \mathcal{R}_{0+}} \partial^c \mathcal{A}_j(\hat{a}),$$
$$\mathcal{V}_{\mathcal{A}_2} := \bigcup_{j \in \mathcal{R}_{00} \cup \mathcal{R}_{0-}} -\partial^c \mathcal{A}_j(\hat{a}), \quad \mathcal{V}_{\mathcal{B}_1} := \bigcup_{j \in \mathcal{R}_{+0}} \partial^c \mathcal{B}_j(\hat{a}), \quad \mathcal{V}_{\mathcal{B}_2} := \bigcup_{j \in \mathcal{R}_{+0} \cup \mathcal{R}_{00}} \partial^c \mathcal{B}_j(\hat{a}).$$

The following theorem from Upadhyay et al. [24] will be useful in the sequel.

Theorem 3.6. *Let $\hat{a} \in \mathcal{F}$ be a weak Pareto efficient solution of NSIMOPVC at which (NSIMOPVC-ACQ) holds. If the set*

$$\mathcal{U}_1 := \text{pos}\left(\mathcal{V}_\Psi \cup \mathcal{V}_{M_2} \cup \mathcal{V}_{N_2}\right) + \text{span}\left(\mathcal{V}_\theta \cup \mathcal{V}_{M_1}\right)$$

is closed, then $\hat{a}$ is a strong stationary element of NSIMOPVC.

Now, we formulate Lagrange-type dual problems related to NSIMOPVC. Furthermore, we deduce the weak and strong duality relations that relate the primal problem NSIMOPVC and the corresponding Lagrange dual problems.
Let us define the following single-valued scalarized dual map:

$$\mathcal{P}(\alpha, \sigma) = \min_{y \in \mathcal{F}} \mathcal{L}(y, \alpha, \sigma),$$

where, the function $\mathcal{L}(y, \alpha, \sigma)$ is the Lagrangian function associated to NSIMOPVC, as defined below:

$$\mathcal{L}(y, \alpha, \sigma) := \sum_{i \in I} \alpha_i f_i(y) + \sum_{t \in \mathcal{T}} \sigma_t^\Psi \Psi_t(y) + \sum_{j \in \mathcal{J}} \sigma_j^\theta \theta_j(y) - \sum_{j \in \mathcal{S}} \left(\sigma_j^\mathcal{A} \mathcal{A}_j(y) - \sigma_j^\mathcal{B} \mathcal{B}_j(y)\right),$$

where $\alpha \in \mathbb{R}_+^m$ is fixed with $\sum_{j=1}^{m} \alpha_j = 1$, and $\sigma = (\sigma^\Psi, \sigma^\theta, \sigma^\mathcal{A}, \sigma^\mathcal{B}) \in \mathbb{R}_+^{|\mathcal{T}|} \times \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^p$.
Let $\hat{a} \in \mathcal{F}$ be any arbitrary feasible element of NSIMOPVC. Then the Lagrange-type dual model depending on $\hat{a}$ (in short, LDP-($\hat{a}, \alpha$) related to NSIMOPVC, is formulated in the following manner:

$$\begin{aligned} (\text{LDP-}(\hat{a}, \alpha)) \qquad & \text{Maximize } \mathcal{P}(\alpha, \sigma), \\ \text{subject to} \qquad & \sigma_t^\Psi \geq 0, \ \forall t \in \mathcal{T} \setminus L(\hat{a}), \\ & \sigma_j^\mathcal{A} \geq 0, \ \forall j \in \mathcal{R}_+(\hat{a}), \\ & \sigma_j^\mathcal{B} \geq 0, \ \forall j \in \mathcal{R}_{+-}(\hat{a}) \cup \mathcal{R}_{0-}(\hat{a}), \\ & \sigma_j^\mathcal{B} \leq 0, \ \forall j \in \mathcal{R}_{0+}(\hat{a}). \end{aligned}$$

Let $\mathcal{F}_D(\hat{a}, \alpha)$ denote the feasible set of the problem LDP-($\hat{a}, \alpha$). Now, we formulate another Lagrange-type dual model that does not explicitly depend on the feasible element $\hat{a} \in \mathcal{F}$ as follows:

$$\begin{aligned} (\text{LDP-}(\alpha)) \qquad & \text{Maximize } \mathcal{P}(\alpha, \sigma), \\ \text{subject to} \qquad & \sigma \in \mathcal{F}_D(\alpha) = \bigcap_{\hat{a} \in \mathcal{F}} \mathcal{F}_D(\hat{a}), \end{aligned}$$

where the set $\mathcal{F}_D(\alpha) \neq \emptyset$ is the feasible set of the problem LDP-(α).

Remark 3.7. From the above constructions of the feasible sets of LDP-($\hat{a}$, α) and LDP-(α), it is obvious that every feasible element of the problem LDP-(α) is automatically a feasible element of LDP-($\hat{a}$, α). Moreover, LDP-(α) is independent of any particular choice of $\hat{a} \in \mathcal{F}$. As a result, LDP-(α) serves as a more preferable choice of dual model as compared to LDP-($\hat{a}$, α).

In the following theorem, we derive weak duality relation that relate NSIMOPVC and (LDP-($\hat{a}$, α)).

Theorem 3.8. *Let $\hat{a} \in \mathcal{F}$, $\sigma \in \mathcal{F}_D(\hat{a}, \alpha)$ be arbitrary feasible elements. Then the following inequality is satisfied:*

$$\mathcal{P}(\alpha, \sigma) \leq \sum_{i=1}^{m} f_i \phi_i(\hat{a}).$$

Proof. In light of the provided hypothesis, we have $\hat{a} \in \mathcal{F}$ and $\sigma \in \mathcal{F}_D(\hat{a}, \alpha)$. As a result, it follows that:

$$\mathcal{P}(\alpha, \sigma) = \min_{y \in \mathcal{F}} \mathcal{L}(y, \alpha, \sigma) \leq \sum_{i \in I} \alpha_i f_i(y) + \sum_{t \in \mathcal{T}} \sigma_t^{\Psi} \Psi_t(y) + \sum_{j \in \mathcal{J}} \sigma_j^{\theta} \theta_j(y) - \sum_{j \in \mathcal{S}} \left(\sigma_j^{\mathcal{A}} \mathcal{A}_j(y) - \sigma_j^{\mathcal{B}} \mathcal{B}_j(y) \right). \tag{3.1}$$

As $\hat{a} \in \mathcal{F}$, it follows that:

$$\begin{aligned} \Psi_t(\hat{a}) &\leq 0, \quad \forall t \in \mathcal{T}, \\ \theta_i(\hat{a}) &= 0, \quad \forall i \in \mathcal{J}, \\ -\mathcal{A}_i(\hat{a}) &\leq 0, \quad \forall i \in \mathcal{S}. \end{aligned}$$

Further, it is given that $\sigma \in \mathcal{F}_D(\hat{a}, \alpha)$. Consequently, we have:

$$\sum_{t \in L(\hat{a})} \sigma_t^{\Psi} \Psi_t(\hat{a}) = 0, \quad \sum_{i \in \mathcal{R}_0(\hat{a})} \sigma_i^{\mathcal{A}} \mathcal{A}_i(\hat{a}) = 0, \quad \sum_{i \in \mathcal{R}_{+0}(\hat{a}) \cup \mathcal{R}_{00}(\hat{a})} \sigma_i^{\mathcal{B}} \mathcal{B}_i(\hat{a}) = 0,$$

$$\begin{aligned} \Psi_t(\hat{a}) < 0 \text{ and } \sigma_t^{g} \geq 0, &\quad \forall t \in \mathcal{T} \backslash L(\hat{a}), \\ -\mathcal{A}_i(\hat{a}) < 0 \text{ and } \sigma_i^{\mathcal{A}} \geq 0, &\quad \forall i \in \mathcal{R}_{+}(\hat{a}), \\ \mathcal{B}_i(\hat{a}) > 0 \text{ and } \sigma_i^{\mathcal{B}} \leq 0, &\quad \forall i \in \mathcal{R}_{0+}(\hat{a}), \\ \mathcal{B}_i(\hat{a}) < 0 \text{ and } \sigma_i^{\mathcal{B}} \geq 0, &\quad \forall i \in \mathcal{R}_{+-}(\hat{a}) \cup \mathcal{R}_{0-}(\hat{a}). \end{aligned}$$

In view of the above expressions, one can infer that:

$$\sum_{t \in \mathcal{T}} \sigma_t^{\Psi} \Psi_t(y) + \sum_{j \in \mathcal{J}} \sigma_j^{\theta} \theta_j(y) - \sum_{j \in \mathcal{S}} \left(\sigma_j^{\mathcal{A}} \mathcal{A}_j(y) - \sigma_j^{\mathcal{B}} \mathcal{B}_j(y) \right) \leq 0. \tag{3.2}$$

In view of (3.1) and (3.2), it readily follows that:

$$\mathcal{P}(\alpha, \sigma) \leq \sum_{i=1}^{p} \alpha_i f_i(\hat{a}).$$

Thus the proof is complete. □

Remark 3.9. Theorem 3.8 extends Proposition 4.1 established by Tung et al. [21] from the setting of Euclidean space to the framework of a more general space, namely, Hadamard manifolds.

In the following corollary, we derive weak duality relation that relate NSIMOPVC and (LDP).

Corollary 3.10. *Let $\hat{a} \in \mathcal{F}$, $\sigma \in \mathcal{F}_D(\alpha)$ be arbitrary elements. Then the following inequality is satisfied.*

$$\mathcal{P}(\alpha, \sigma) \leq \sum_{i=1}^{p} \alpha_i f_i(\hat{a}).$$

In the following theorem, we derive strong duality relation that relates NSIMOPVC and LDP-$(\hat{a}, \alpha)$.

Theorem 3.11. *Let $\hat{a}$ be a locally weakly Pareto efficient solution of the NSIMOPVC at which NSIMOPVC-ACQ is satisfied. We suppose that the set $\mathcal{U}_1$ is closed at $\hat{a}$. Moreover, suppose that the following functions:*

$$\begin{aligned} &f_i \ (i \in I), && \Psi_t \ (t \in L^{+}(\hat{a})), \\ &\theta_i \ \left(i \in \mathcal{J}^{+}(\hat{a})\right), && -\theta_i \ (i \in \mathcal{J}^{-}(\hat{a})), \\ &\mathcal{A}_i \ \left(i \in \hat{\mathcal{R}}_{0+}^{-}(\hat{a})\right), && -\mathcal{A}_i \ \left(i \in \hat{\mathcal{R}}_{0+}^{+}(\hat{a}) \cup \hat{\mathcal{R}}_{00}^{+}(\hat{a}) \cup \hat{\mathcal{R}}_{0-}^{+}(\hat{a})\right), \\ &\mathcal{B}_i \ \left(i \in \mathcal{R}_{\pm 0}^{+}(\hat{a}) \cup \mathcal{R}_{00}^{+}(\hat{a})\right), && \end{aligned}$$

be geodesic convex at $\hat{a}$. Then, there exists some $\widetilde{\alpha} \in \mathbb{R}^+_m$, with $\sum_{i\in I} \widetilde{\alpha}_i = 1$, such that $\widetilde{\sigma}$ is an optimal solution of LDP($\hat{a}, \overline{\alpha}$) and

$$\sum_{i\in I} \overline{\alpha}_i f_i(\hat{a}) = \mathcal{P}(\widetilde{\alpha}, \widetilde{\sigma}).$$

Proof. In view of Theorem 3.6, there exist $\widetilde{\alpha} \in \mathbb{R}^m_+$, $\widetilde{\sigma}^\Psi \in \mathscr{A}(\hat{a})$, $\widetilde{\sigma}^\theta \in \mathbb{R}^q$, $\widetilde{\sigma}^{\mathcal{A}} \in \mathbb{R}^p$, $\widetilde{\sigma}^{\mathcal{B}} \in \mathbb{R}^p$, $\xi^f_j \in \partial^c f_j(\hat{a})$, $\xi^\Psi_j \in \partial^c \Psi_j(\hat{a})$, $\xi^\theta_j \in \partial^c \theta_j(\hat{a})$, $\xi^{\mathcal{A}}_j \in \partial^c \mathcal{A}_j(\hat{a})$, $\xi^{\mathcal{B}}_j \in \partial^c \mathcal{B}_j(\hat{a})$, satisfying:

$$0 = \sum_{j\in I} \widetilde{\alpha}_j \xi^f_j + \sum_{t\in\mathcal{T}} \widetilde{\sigma}^\Psi_t \xi^\Psi_t + \sum_{j\in\mathcal{J}} \widetilde{\sigma}^\theta_j \xi^\theta_j - \sum_{j\in\mathcal{S}} \widetilde{\sigma}^{\mathcal{A}}_j \xi^{\mathcal{A}}_j + \sum_{j\in\mathcal{S}} \widetilde{\sigma}^{\mathcal{B}}_j \xi^{\mathcal{B}}_j,$$

$$\begin{aligned}
&\widetilde{\sigma}^{\mathcal{A}}_j = 0, \quad \forall j \in \mathcal{R}_+(\hat{a}), \quad \widetilde{\sigma}^{\mathcal{A}}_j \geq 0, \quad \forall j \in \mathcal{R}_{00}(\hat{a}) \cup \mathcal{R}_{0-}(\hat{a}),\\
&\widetilde{\sigma}^{\mathcal{B}}_j = 0, \quad \forall j \in \mathcal{R}_{+-}(\hat{a}) \cup \mathcal{R}_{0+}(\hat{a}) \cup \mathcal{R}_{0-}(\hat{a}),\\
&\widetilde{\sigma}^{\mathcal{B}}_j \geq 0, \quad \forall j \in \mathcal{R}_{+0}(\hat{a}) \cup \mathcal{R}_{00}(\hat{a}), \quad \text{and} \sum_{j\in I} \widetilde{\alpha}_j = 1.
\end{aligned}$$

From the provided hypothesis, we have that $\widetilde{\sigma}^\Psi \in \mathscr{A}(\hat{a})$. As a result, we have the following:

$$\widetilde{\sigma}^\Psi_t \Psi_t(\hat{a}) = 0, \quad \forall t \in \mathcal{T}, \quad \text{and} \sum_{t\in\mathcal{T}} \widetilde{\sigma}^\Psi_t \Psi_t(\hat{a}) = 0.$$

On the other hand, it is clear that

$$\Psi_t(\hat{a}) < 0, \ \forall t \in \mathcal{T} \backslash L(\hat{a}).$$

This entails the following:

$$\begin{aligned}
&\widetilde{\sigma}^\Psi_t = 0, \ \forall t \in \mathcal{T} \backslash L(\hat{a}),\\
&\widetilde{\sigma} \in \mathcal{F}_D(\hat{a}, \overline{\alpha}).
\end{aligned}$$

Similarly, from the feasibility conditions of NSIMOPVC, we have:

$$\sum_{i\in\mathcal{J}} \widetilde{\sigma}^\theta_i \theta_i(\hat{a}) = 0.$$

It can be easily noticed that:

$$\widetilde{\sigma}^{\mathcal{A}}_{\mathcal{R}_+(\hat{a})} = 0, \quad \mathcal{A}_i(\hat{a}) = 0, \quad \forall i \in \mathcal{R}_0(\hat{a}).$$

Hence, we infer the following:

$$\sum_{i\in\mathcal{S}} \widetilde{\sigma}^{\mathcal{A}}_i \mathcal{A}_i(\hat{a}) = 0.$$

Proceeding in similar manner, we obtain:

$$\sum_{i\in\mathcal{S}} \widetilde{\sigma}^{\mathcal{B}}_i \mathcal{B}_i(\hat{a}) = 0.$$

Therefore, we arrive at the following:

$$\sum_{t\in\mathcal{T}} \widetilde{\sigma}^\Psi_t \Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \widetilde{\sigma}^\theta_i \theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \widetilde{\sigma}^{\mathcal{A}}_i \mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \widetilde{\sigma}^{\mathcal{B}}_i \mathcal{B}_i(\hat{a}) = 0.$$

This, in turn, concludes the following:

$$\sum_{i\in I} \widetilde{\alpha}_i f_i(\hat{a}) = \mathcal{L}(\hat{a}, \widetilde{\alpha}, \widetilde{\sigma}). \tag{3.3}$$

In light of the geodesic convexity assumptions on the objective function and constraints, we obtain that the following inequalities are satisfied for every $a \in \mathcal{F}$:

$$\begin{aligned}
f_j(a) - f_j(\hat{a}) &\geq \langle \xi_i^f, \exp_{\hat{a}}^{-1}(a)\rangle, && \forall j \in I,\ \xi_i^f \in \partial^c f_j(\hat{a}),\\
\Psi_t(a) - \Psi_t(\hat{a}) &\geq \langle \xi_t^\Psi, \exp_{\hat{a}}^{-1}(a)\rangle, && \forall t \in L^+(\hat{a}),\ \xi_t^\Psi \in \partial^c \Psi_t(\hat{a}),\\
\theta_j(a) - \theta_j(\hat{a}) &\geq \langle \xi_j^\theta, \exp_{\hat{a}}^{-1}(a)\rangle, && \forall j \in \mathcal{J}^+(\hat{a}),\ \xi_j^\theta \in \partial^c \theta_j(\hat{a}),\\
-\theta_j(a) + \theta_j(\hat{a}) &\geq \langle -\xi_j^\theta, \exp_{\hat{a}}^{-1}(a)\rangle, && \forall j \in \mathcal{J}^-(\hat{a}),\ \xi_j^\theta \in \partial^c \theta_j(\hat{a}),\\
\mathcal{A}_j(a) - \mathcal{A}_j(\hat{a}) &\geq \left\langle \xi_j^\mathcal{A}, \exp_{\hat{a}}^{-1}(a)\right\rangle, && \forall j \in \hat{\mathcal{R}}_{0+}^-(\hat{a}),\ \xi_j^\mathcal{A} \in \partial^c \mathcal{A}_j(\hat{a}),\\
\mathcal{B}_j(a) - \mathcal{B}_j(\hat{a}) &\geq \left\langle \xi_j^\mathcal{B}, \exp_{\hat{a}}^{-1}(a)\right\rangle, && \forall j \in \mathcal{R}_{+0}^+(\hat{a}) \cup \mathcal{R}_{00}^+,\ \xi_j^\mathcal{B} \in \partial^c \mathcal{B}_j(\hat{a}),\\
-\mathcal{A}_j(a) + \mathcal{A}_j(\hat{a}) &\geq -\left\langle \xi_j^\mathcal{A}, \exp_{\hat{a}}^{-1}(a)\right\rangle, && \forall j \in \hat{\mathcal{R}}_{0+}^+(\hat{a}) \cup \hat{\mathcal{R}}_{00}^+(\hat{a}) \cup \hat{\mathcal{R}}_{0-}^+(\hat{a}),\\
& && \xi_j^\mathcal{A} \in \partial^c \mathcal{A}_j(\hat{a}).
\end{aligned} \tag{3.4}$$

As a result, it follows that:

$$\begin{aligned}
&\sum_{i\in I} \widetilde{\alpha}_i f_i(a) + \sum_{t\in\mathcal{T}} \widetilde{\sigma}_t^\Psi \Psi_t(a) + \sum_{i\in\mathcal{J}} \widetilde{\sigma}_i^\theta \theta_i(a) - \sum_{i\in\mathcal{S}} \widetilde{\sigma}_i^H \mathcal{A}_i(a) + \sum_{i\in\mathcal{S}} \widetilde{\sigma}_i^\mathcal{B} \mathcal{B}_i(a)\\
&-\left(\sum_{i\in I} \widetilde{\alpha}_i f_i(\hat{a}) + \sum_{t\in\mathcal{T}} \widetilde{\sigma}_t^\Psi \Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \widetilde{\sigma}_i^\theta \theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \widetilde{\sigma}_i^H \mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \widetilde{\sigma}_i^\mathcal{B} \mathcal{B}_i(\hat{a})\right)\\
&\geq \Big\langle \sum_{i\in I} \widetilde{\alpha}_i \xi_i^f + \sum_{t\in\mathcal{T}} \widetilde{\sigma}_t^\Psi \xi_t^\Psi + \sum_{i\in\mathcal{J}} \widetilde{\sigma}_i^\theta \xi_i^\theta\\
&\qquad\qquad - \sum_{i\in\mathcal{S}} \widetilde{\sigma}_i^H \xi_i^\mathcal{A} + \sum_{i\in\mathcal{S}} \widetilde{\sigma}_i^\mathcal{B} \xi_i^\mathcal{B},\ \exp_{\hat{a}}^{-1}(a)\Big\rangle = 0.
\end{aligned} \tag{3.5}$$

Thus, we arrive at the the following inequality:

$$\begin{aligned}
&\sum_{i=1}^p \widetilde{\alpha}_i f_i(\hat{a}) + \sum_{k=1}^r \sigma_t^\Psi \Psi_t(\hat{a}) + \sum_{j=1}^q \sigma_j^\theta \theta_j(\hat{a}) - \sum_{j=1}^m [\sigma_j^\mathcal{A} \mathcal{A}_j(\hat{a}) + \sigma_j^\mathcal{B} \mathcal{B}_j(\hat{a})]\\
\leq &\sum_{i=1}^p \widetilde{\alpha}_i f_i(\hat{a}) + \sum_{k=1}^r \widetilde{\sigma}_t^\Psi \Psi_t(\hat{a}) + \sum_{j=1}^q \widetilde{\sigma}_j^\theta \theta_j(\hat{a}) - \sum_{j=1}^m [\widetilde{\sigma}_j^P \mathcal{A}_j(\hat{a} + \widetilde{\sigma}_j \mathcal{B}_j(\hat{a})].
\end{aligned}$$

Thus, it follows that

$$\mathcal{L}(\hat{a}, \widetilde{\alpha}, \sigma) \leq \mathcal{L}(a, \widetilde{\alpha}, \widetilde{\sigma}),\ \forall a \in \mathcal{F}. \tag{3.6}$$

Therefore, we get

$$\sum_{i=1}^p \widetilde{\alpha}_i f_i(\hat{a}) = \mathcal{L}(\widetilde{\alpha}, \hat{a}, \widetilde{\sigma}) = \mathcal{P}(\widetilde{\alpha}, \widetilde{\sigma}). \tag{3.7}$$

Again, by weak duality theorem, we get

$$\mathcal{P}(\widetilde{\alpha}, \sigma) \leq \sum_{i=1}^p \widetilde{\alpha}_i f_i(\hat{a}), \quad \forall \sigma \in \mathcal{F}_D(\hat{a}). \tag{3.8}$$

From (3.7) and (3.8), we get

$$\mathcal{P}(\widetilde{\alpha}, \sigma) \leq \mathcal{P}(\widetilde{\alpha}, \widetilde{\sigma}),\ \forall \sigma \in \mathcal{F}_D(\hat{a}). \tag{3.9}$$

This completes the proof. □

Remark 3.12. Theorem 3.11 extends Proposition 4.4 established by Tung et al. [21] from the setting of Euclidean space to the framework of a more general space, namely, Hadamard manifolds.

In the following definition, we recall the notion of saddle point of NSIMOPVC (see, for instance, [21]).

Definition 3.13. Suppose that $\widetilde{\alpha} \in \mathbb{R}^m_+$ be a fixed vector, satisfying

$$\widetilde{\alpha}_1 + \widetilde{\alpha}_2 + \ldots + \widetilde{\alpha}_m = 1.$$

Let $\hat{a} \in \mathcal{F}$ and $\widetilde{\sigma} \in \mathcal{F}_D(\hat{a}, \widetilde{\alpha})$. Then, $(\hat{a}, \widetilde{\sigma})$ is referred to as a saddle point of NSIMOPVC if the following inequalities are satisfied:

$$\text{(i) } \mathcal{L}(\hat{a}, \widetilde{\alpha}, \sigma) \leq \mathcal{L}(\hat{a}, \widetilde{\alpha}, \widetilde{\sigma}),$$
$$\text{(ii) } \mathcal{L}(\hat{a}, \widetilde{\alpha}, \widetilde{\sigma}) \leq \mathcal{L}(a, \widetilde{\alpha}, \widetilde{\sigma}).$$

for every $a \in \mathcal{F}$ and $\sigma \in \mathcal{F}_D(\hat{a}, \widetilde{\alpha})$.

In the following theorem, we derive a sufficient saddle point optimality condition for NSIMOPVC.

Theorem 3.14. *Let $\hat{a}$ be a locally weakly Pareto efficient solution of the NSIMOPVC at which NSIMOPVC-ACQ is satisfied. We suppose that the set $\mathcal{U}_1$ is closed at $\hat{a}$. Moreover, suppose that the following functions:*

$$\begin{aligned}
&f_i \ (i \in I), && \Psi_t \ (t \in L^+(\hat{a})),\\
&\theta_i \ \left(i \in \mathcal{J}^+(\hat{a})\right), && -\theta_i \ (i \in \mathcal{J}^-(\hat{a})),\\
&\mathcal{A}_i \ \left(i \in \hat{\mathcal{R}}^-_{0+}(\hat{a})\right), && -\mathcal{A}_i \ \left(i \in \hat{\mathcal{R}}^+_{0+}(\hat{a}) \cup \hat{\mathcal{R}}^+_{00}(\hat{a}) \cup \hat{\mathcal{R}}^+_{0-}(\hat{a})\right),\\
&\mathcal{B}_i \ \left(i \in \mathcal{R}^+_{\pm 0}(\hat{a}) \cup \mathcal{R}^+_{00}(\hat{a})\right),
\end{aligned}$$

be geodesic convex at $\hat{a}$. Then, there exist $\widetilde{\sigma} \in \mathscr{A}(\hat{a}) \times \mathbb{R}^q_+ \times \mathbb{R}^p \times \mathbb{R}^p$, such that $(\hat{a}, \widetilde{\sigma})$ is a saddle point of NSIMOPVC.

Proof. In view of (3.6), we have:

$$\mathcal{L}(\hat{a}, \widetilde{\alpha}, \sigma) \leq \mathcal{L}(a, \widetilde{\alpha}, \widetilde{\sigma}), \ \forall a \in \mathcal{F}. \tag{3.10}$$

Moreover, for any $\sigma \in \mathcal{F}_D(\hat{a}, \widetilde{\sigma})$, we have:

$$\begin{aligned}
\mathcal{L}(\hat{a}, \widetilde{\alpha}, \widetilde{\sigma}) &= \sum_{i=1}^{p} \widetilde{\alpha}_i f_i(\hat{a})\\
&\geq \sum_{i=1}^{p} \widetilde{\alpha}_i f_i(\hat{a}) + \sum_{k=1}^{r} \sigma_t^{\Psi} \Psi_t(\hat{a}) + \sum_{j=1}^{q} \sigma_j^{\theta} \theta_j(\hat{a}) - \sum_{j=1}^{m} \left[\sigma_j^{\mathcal{A}} \mathcal{A}_j(\hat{a}) + \sigma_j^{\mathcal{B}} \mathcal{B}_j(\hat{a})\right].
\end{aligned}$$

Thus we have:

$$\mathcal{L}(\hat{a}, \widetilde{\alpha}, \sigma) \leq \mathcal{L}(\hat{a}, \widetilde{\alpha}, \widetilde{\sigma}), \ \forall \sigma \in \mathcal{F}_D(\hat{a}, \widetilde{\sigma}).$$

This completes the proof. □

4. Vector Lagrange-Type Duality and Saddle Point Optimality for NSIMOPVC

In the present section, corresponding to the considered problem NSIMOPVC, we formulate the corresponding weak vector Lagrange-type dual model. Subsequently, we establish weak and strong duality results that relate the primal-dual pair.

For any $a \in \mathcal{F}$, the corresponding vector Lagrangian of NSIMOPVC, denoted by $\overline{\mathcal{L}}(a, \sigma)$, is defined as follows:

$$\overline{\mathcal{L}}(a, \sigma) := f(a) + \left(\sum_{t \in \mathcal{T}} \sigma_t^{\Psi} \Psi_t(a) + \sum_{j \in \mathcal{J}} \sigma_j^{\theta} \theta_j(a) - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{A}} \mathcal{A}_j(a) + \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{B}} \mathcal{B}_j(a)\right) e,$$

where $\sigma := \left(\sigma^{\Psi}, \sigma^{\theta}, \sigma^{\mathcal{A}}, \sigma^{\mathcal{B}}\right) \in \mathbb{R}_{+}^{|\mathcal{T}|} \times \mathbb{R}^{q} \times \mathbb{R}^{p} \times \mathbb{R}^{p}$ and $e := (1, \ldots, 1) \in \mathbb{R}^{m}$. Moreover, we denote the set-valued weak dual map $\omega^{W} : \mathbb{R}^{m} \rightrightarrows \mathbb{R}^{m}$ as:

$$\omega^{W}(\sigma) := \underset{\mathbb{R}_{+}^{m}}{\mathrm{WMin}} \left\{\overline{\mathcal{L}}(a, \sigma) \mid a \in \mathcal{F}\right\}.$$

The weak vector Lagrange-type dual model related to NSIMOPVC, depending on the choice of some $a \in \mathcal{F}$, may be formulated in the following manner:

$$\begin{aligned} (\mathrm{DV}(a)) \quad & \mathbb{R}_{+}^{m} - \mathrm{WMax}\, \Phi^{W}(\sigma), \\ \text{subject to} \quad & \sigma_{j}^{\Psi} \geq 0, \ \forall j \in \mathcal{T} \backslash L(a), \\ & \sigma_{j}^{\mathcal{A}} \leq 0, \ \forall j \in \mathcal{R}_{0+}(a), \\ & \sigma_{j}^{\mathcal{A}} \geq 0, \ \forall j \in \mathcal{R}_{+-}(a) \cup \mathcal{R}_{0-}(a), \\ & \sigma_{j}^{\mathcal{B}} \geq 0, \ \forall j \in \mathcal{R}_{+}(a). \end{aligned}$$

The set of all feasible points of the problem $\mathrm{DV}(a)$ is signified by $\mathcal{F}_{1}(a)$. Now, we recall the notion of weakly Pareto efficient solution of DV(a) (see, for instance, [21]).

Definition 4.1. Let $\overline{a} \in \bigcup_{\sigma \in \mathcal{F}_{1}(a)} \omega^{W}(\sigma)$ be arbitrary. Then, $\overline{a}$ is termed as a weakly Pareto efficient solution of $\mathrm{DV}(a)$, provided that:

$$\overline{a} \in \underset{\mathbb{R}_{+}^{m}}{\mathrm{WMax}} \bigcup_{\sigma \in \mathcal{F}_{1}(a)} \Phi^{W}(\sigma).$$

Equivalently, this implies that there does not exist any $\hat{a} \in \bigcup_{\sigma \in \mathcal{F}_{1}(a)} \omega^{W}(\sigma)$, such that

$$\overline{a} \prec \hat{a}.$$

In a similar manner as before, we now formulate the Lagrange-type dual model which is independent on the choice of a feasible element of NSIMOPVC:

$$\begin{aligned} (\mathrm{DV}) \quad & \mathbb{R}_{+}^{m} - \mathrm{WMax}\, \omega^{W}(\sigma), \\ \text{subject to} \quad & \sigma \in \mathcal{F}_{1} := \bigcap_{a \in \mathcal{F}} \mathcal{F}_{1}(a), \end{aligned}$$

where $\mathcal{F}_{1} := \bigcap_{a \in \mathcal{F}} \mathcal{F}_{1}(a) \neq \emptyset$ represents the feasible set of the problem DV.

In the following theorem, we establish weak duality relation NSIMOPVC and VD($\hat{a}$), where $\hat{a} \in \mathcal{F}$.

Theorem 4.2. *Let $\hat{a} \in \mathcal{F}$ be any feasible element of NSIMOPVC and let*

$$z \in \bigcup_{\sigma \in \mathcal{F}_{1}(\hat{a})} \Phi^{W}(\sigma).$$

Then, the following holds:

$$f(\hat{a}) \nprec z.$$

Proof. It is given that $\hat{a} \in \mathcal{F}$ and $z \in \bigcup_{\sigma \in \mathcal{F}_{1}(\hat{a})} \Phi^{W}(\sigma)$. Consequently, there exists some $\sigma \in \mathcal{F}_{1}(\hat{a})$, such that $z \in \Phi^{W}(\sigma)$. As a result, we have the following:

$$f(\hat{a}) + \left(\sum_{t \in \mathcal{T}} \sigma_{t}^{\Psi} \Psi_{t}(\hat{a}) + \sum_{j \in \mathcal{J}} \sigma_{j}^{\theta} \theta_{j}(\hat{a}) + \sum_{j \in \mathcal{S}} \sigma_{j}^{\mathcal{B}} \mathcal{B}_{j}(\hat{a}) - \sum_{j \in \mathcal{S}} \sigma_{j}^{\mathcal{A}} \mathcal{A}_{j}(\hat{a})\right) e \nprec z.$$

By *reductio ad absurdum*, let us assume that

$$f(\hat{a}) \prec z. \tag{4.1}$$

From the feasibility conditions of NSIMOPVC, we have the following:

$$\begin{aligned} \Psi_t(\hat{a}) &\leq 0, \quad \forall t \in \mathcal{T}, \\ \theta_j(\hat{a}) &= 0, \quad \forall j \in \mathcal{J}, \\ \mathcal{A}_j(\hat{a}) &\geq 0, \quad \forall j \in \mathcal{S}. \end{aligned}$$

Further, we have $\sigma \in \mathcal{F}_1(\hat{a})$. Evidently, we arrive at the following:

$$\sum_{t \in L(\hat{a})} \sigma_t^{\Psi} \Psi_t(\hat{a}) = 0, \quad \sum_{j \in \mathcal{R}_0(\hat{a})} \sigma_j^{\mathcal{A}} \mathcal{A}_j(\hat{a}) = 0, \quad \sum_{j \in \mathcal{R}_{+0}(\hat{a}) \cup \mathcal{R}_{00}(\hat{a})} \sigma_j^{\mathcal{B}} \mathcal{B}_j(\hat{a}) = 0,$$

$$\begin{aligned} \Psi_t(\hat{a}) < 0, \ \sigma_t^{\Psi} \geq 0, \ \forall t \in \mathcal{T} \setminus L(\hat{a}), \\ \mathcal{B}_j(\hat{a}) < 0, \ \sigma_j^{\mathcal{B}} \geq 0, \ \forall j \in \mathcal{R}_{+-}(\hat{a}) \cup \mathcal{R}_{0-}(\hat{a}), \\ -\mathcal{A}_j(\hat{a}) < 0, \ \sigma_j^{\mathcal{A}} \geq 0, \ \forall j \in \mathcal{R}_{+}(\hat{a}), \\ \mathcal{B}_j(\hat{a}) > 0, \ \sigma_j^{\mathcal{B}} \leq 0, \ \forall j \in \mathcal{R}_{0+}(\hat{a}), \end{aligned}$$

Hence for all $j \in I$, it follows that:

$$\sum_{t \in \mathcal{T}} \sigma_t^{\Psi} \Psi_t(\hat{a}) + \sum_{j \in \mathcal{J}} \sigma_j^{\theta} \theta_j(\hat{a}) + \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{B}} \mathcal{B}_j(\hat{a}) - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{A}} \mathcal{A}_j(\hat{a}) \leq 0. \tag{4.2}$$

Evidently, for all $j \in I$, we have the following:

$$f_j(\hat{a}) + \left(\sum_{t \in \mathcal{T}} \sigma_t^{\Psi} \Psi_t(\hat{a}) + \sum_{j \in \mathcal{J}} \sigma_j^{\theta} \theta_j(\hat{a}) + \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{B}} \mathcal{B}_j(\hat{a}) - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{A}} \mathcal{A}_j(\hat{a}) \right) \leq f_j(\hat{a}).$$

This implies that

$$f(\hat{a}) + \left(\sum_{t \in \mathcal{T}} \sigma_t^{\Psi} \Psi_t(\hat{a}) + \sum_{j \in \mathcal{J}} \sigma_j^{\theta} \theta_j(\hat{a}) + \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{B}} \mathcal{B}_j(\hat{a}) - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{A}} \mathcal{A}_j(\hat{a}) \right) e. \preccurlyeq f(\hat{a}) \tag{4.3}$$

In view of the above inequality, it follows that

$$f(\hat{a}) + \left(\sum_{t \in \mathcal{T}} \sigma_t^{\Psi} \Psi_t(\hat{a}) + \sum_{j \in \mathcal{J}} \sigma_j^{\theta} \theta_j(\hat{a}) + \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{B}} \mathcal{B}_j(\hat{a}) - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{A}} \mathcal{A}_j(\hat{a}) \right) e \prec z,$$

which is a contradiction. Thus the proof is complete. □

Remark 4.3. Theorem 4.2 extends Proposition 3.1 established by Tung et al. [21] from the setting of Euclidean space to the framework of a more general space, namely, Hadamard manifolds.

Theorem 4.4. *Let us assume that* $\hat{a} \in \mathcal{F}$ *and* $\overline{\sigma} \in \mathcal{F}_{DV(\hat{a})}$. *Further, we suppose that*

$$f(\hat{a}) \in \omega^W(\overline{\sigma}).$$

Then, $\overline{z} = f(\hat{a})$ *is a weakly Pareto efficient element of DV(*$\hat{a}$*). Furthermore, let* $\overline{\sigma} \in \mathcal{F}_1$. *Then,* $\hat{a}$ *is a weakly Pareto efficient solution of NSIMOPVC.*

Proof. By *reductio ad absurdum*, we assume that $\overline{z} = f(\hat{a})$ is not a weakly Pareto efficient element of DV($\hat{a}$). Consequently, some $\overline{\sigma} \in \mathcal{F}_1(\hat{a})$ and $z \in \omega^W(\overline{\sigma})$ exist, such that

$$\overline{y} = f(\hat{a}) \prec z. \tag{4.4}$$

In view of the facts that $\hat{a} \in \mathcal{F}$ and $\overline{\sigma} \in \mathcal{F}_{DV(\hat{a})}$, we conclude:

$$f(\hat{a}) + \left(\sum_{t \in \mathcal{T}} \overline{\sigma}_t^{\Psi} \Psi_t(\hat{a}) + \sum_{j \in \mathcal{J}} \overline{\sigma}_j^{\theta} \theta_j(\hat{a}) + \sum_{j \in \mathcal{S}} \overline{\sigma}_j^{\mathcal{B}} \mathcal{B}_j(\hat{a}) - \sum_{j \in \mathcal{S}} \overline{\sigma}_j^{\mathcal{A}} \mathcal{A}_j(\hat{a}) \right) e \preccurlyeq f(\hat{a}) = \overline{z}.$$

Evidently, we have:

$$f(\hat{a}) + \left(\sum_{t \in \mathcal{T}} \overline{\sigma}_t^{\Psi} \Psi_t(\hat{a}) + \sum_{j \in \mathcal{J}} \overline{\sigma}_j^{\theta} \theta_j(\hat{a}) + \sum_{j \in \mathcal{S}} \overline{\sigma}_j^{\mathcal{B}} \mathcal{B}_j(\hat{a}) - \sum_{j \in \mathcal{S}} \overline{\sigma}_j^{\mathcal{A}} \mathcal{A}_j(\hat{a}) \right) e \prec z.$$

This clearly results in a contradiction. Therefore, $\overline{z} = f(\hat{a})$ is a weakly Pareto efficient solution of DV($\hat{a}$).

Again, by *reductio ad absurdum*, we assume that $\hat{a}$ is not a weakly Pareto efficient solutiuon of NSIMOPVC. As a result, we are guaranteed of the existence of some $a \in \mathcal{F}$, such that

$$f(a) \prec f(\hat{a}). \tag{4.5}$$

Continuing in similar manner as done in the proof of Theorem 4.2, one can derive the following:

$$\left(\sum_{t \in \mathcal{T}} \overline{\sigma}_t^{\Psi} \Psi_t(a) + \sum_{j \in \mathcal{J}} \overline{\sigma}_j^{\theta} \theta_j(a) + \sum_{j \in \mathcal{S}} \overline{\sigma}_j^{\mathcal{B}} \mathcal{B}_j(a) - \sum_{j \in \mathcal{S}} \overline{\sigma}_j^{\mathcal{A}} \mathcal{A}_j(a) \right) e \preccurlyeq 0$$

In light of the inequality (4.5), we have:

$$\overline{\mathcal{L}}(a, \overline{\sigma}) = f(a) + \left(\sum_{t \in \mathcal{T}} \overline{\sigma}_t^{\Psi} \Psi_t(a) + \sum_{j \in \mathcal{J}} \overline{\sigma}_j^{\theta} \theta_j(a) + \sum_{j \in \mathcal{S}} \overline{\sigma}_j^{\mathcal{B}} \mathcal{B}_j(a) - \sum_{j \in \mathcal{S}} \overline{\sigma}_j^{\mathcal{A}} \mathcal{A}_j(a) \right) e \prec f(\hat{a}).$$

Hence, we conclude that $\hat{a}$ is a weakly Pareto efficient solution of NSIMOPVC. This completes the proof. □

Remark 4.5. Theorem 3.11 extends Proposition 3.2 established by Tung et al. [21] from the setting of Euclidean space to the framework of a more general space, namely, Hadamard manifolds.

In the following theorem, strong duality relation relating the primal problem NSIMOPVC and the dual problem DV($\hat{a}$) is established in the Hadamard manifold framework.

Theorem 4.6. *Let $\hat{a}$ be a locally weakly Pareto efficient solution of the NSIMOPVC at which NSIMOPVC-ACQ is satisfied. We suppose that the set $\mathcal{U}_1$ is closed at $\hat{a}$. Moreover, suppose that the following functions:*

$$\begin{aligned}
&f_i \ (i \in I), && \Psi_t \ (t \in L^+(\hat{a})),\\
&\theta_i \ \left(i \in \mathcal{J}^+(\hat{a})\right), && -\theta_i \ (i \in \mathcal{J}^-(\hat{a})),\\
&\mathcal{A}_i \ \left(i \in \hat{\mathcal{R}}_{0+}^-(\hat{a})\right), && -\mathcal{A}_i \ \left(i \in \hat{\mathcal{R}}_{0+}^+(\hat{a}) \cup \hat{\mathcal{R}}_{00}^+(\hat{a}) \cup \hat{\mathcal{R}}_{0-}^+(\hat{a})\right),\\
&\mathcal{B}_i \ \left(i \in \mathcal{R}_{\pm 0}^+(\hat{a}) \cup \mathcal{R}_{00}^+(\hat{a})\right), &&
\end{aligned}$$

be geodesic convex at $\hat{a}$. Then there exists $\overline{\sigma} \in \mathcal{F}_1(\hat{a})$, such that $f(\hat{a}) \in \omega^W(\overline{\sigma})$, and therefore, $f(\hat{a})$ is a weakly Pareto efficient solution of DV($\hat{a}$).

Proof. It is given that $\hat{a} \in \mathcal{F}$ is a locally weakly Pareto efficient solution of NSIMOPVC at which NSIMOPVC-ACQ holds. Moreover, it is given that $\mathcal{U}_1$ is closed. Consequently, in view of Theorem

3.6, there exist $\overline{\alpha} \in \mathbb{R}^m_+$, $\overline{\sigma}^\Psi \in \mathscr{A}(\hat{a})$, $\overline{\sigma}^\theta \in \mathbb{R}^q$, $\overline{\sigma}^\mathcal{A} \in \mathbb{R}^p$, $\overline{\sigma}^\mathcal{B} \in \mathbb{R}^p$, $\xi^f_j \in \partial^c f_j(\hat{a})$, $\xi^\Psi_j \in \partial^c \Psi_j(\hat{a})$, $\xi^\theta_j \in \partial^c \theta_j(\hat{a})$, $\xi^\mathcal{A}_j \in \partial^c \mathcal{A}_j(\hat{a})$, $\xi^\mathcal{B}_j \in \partial^c \mathcal{B}_j(\hat{a})$, satisfying:

$$0 = \sum_{j\in I} \overline{\alpha}_j \xi^f_j + \sum_{t\in\mathcal{T}} \overline{\sigma}^\Psi_t \xi^\Psi_t + \sum_{j\in\mathcal{J}} \overline{\sigma}^\theta_j \xi^\theta_j - \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{A}_j \xi^\mathcal{A}_j + \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{B}_j \xi^\mathcal{B}_j,$$

$$\begin{aligned}
&\overline{\sigma}^\mathcal{A}_j = 0, \quad \forall j \in \mathcal{R}_+(\hat{a}), \quad \overline{\sigma}^\mathcal{A}_j \geq 0, \quad \forall j \in \mathcal{R}_{00}(\hat{a}) \cup \mathcal{R}_{0-}(\hat{a}),\\
&\overline{\sigma}^\mathcal{B}_j = 0, \quad \forall j \in \mathcal{R}_{+-}(\hat{a}) \cup \mathcal{R}_{0+}(\hat{a}) \cup \mathcal{R}_{0-}(\hat{a}),\\
&\overline{\sigma}^\mathcal{B}_j \geq 0, \quad \forall j \in \mathcal{R}_{+0}(\hat{a}) \cup \mathcal{R}_{00}(\hat{a}), \quad \text{and} \sum_{j\in I} \overline{\alpha}_j = 1.
\end{aligned}$$

From the definition of the index sets, it clearly follows that:

$$\sum_{t\in\mathcal{T}} \overline{\sigma}^\Psi_t \Psi_t(\hat{a}) + \sum_{j\in\mathcal{J}} \overline{\sigma}^\theta_j \theta_j(\hat{a}) + \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{B}_j \mathcal{B}_j(\hat{a}) - \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{A}_j \mathcal{A}_j(\hat{a}) = 0.$$

As a result, we have the following:

$$f(\hat{a}) = \overline{\mathcal{L}}(\hat{a}, \overline{\sigma}) \tag{4.6}$$

We now claim that:

$$\overline{\mathcal{L}}(a, \overline{\sigma}) \not< \overline{\mathcal{L}}(\hat{a}, \overline{\sigma}), \quad \forall a \in \mathcal{F} \tag{4.7}$$

By *reductio ad absurdum*, we suppose that (4.7) is not valid. Then, there exists some $\widetilde{u} \in \mathcal{F}$, satisfying

$$\left(\overline{\mathcal{L}}\right)_j (\widetilde{u}, \overline{\sigma}) < \left(\overline{\mathcal{L}}\right)_j (\hat{a}, \overline{\sigma}), \text{ for all } j \in I.$$

Equivalently, for every $j \in I$, we have the following:

$$\begin{aligned}
&f_j(\widetilde{u}) + \left(\sum_{t\in\mathcal{T}} \overline{\sigma}^\Psi_t \Psi_t(\widetilde{u}) + \sum_{j\in\mathcal{J}} \overline{\sigma}^\theta_j \theta_j(\widetilde{u}) + \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{B}_j \mathcal{B}_j(\widetilde{u}) - \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{A}_j \mathcal{A}_j(\widetilde{u})\right)\\
<&f_j(\hat{a}) + \left(\sum_{t\in\mathcal{T}} \overline{\sigma}^\Psi_t \Psi_t(\hat{a}) + \sum_{j\in\mathcal{J}} \overline{\sigma}^\theta_j \theta_j(\hat{a}) + \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{B}_j \mathcal{B}_j(\hat{a}) - \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{A}_j \mathcal{A}_j(\hat{a})\right).
\end{aligned}$$

Let us now multiply both sides of the above inequalities with $\overline{\alpha}_j$ and add them. In view of the fact that $\overline{\alpha} \in \mathbb{R}^m_+$ and $\sum_{j\in I} \overline{\alpha}_j = 1$, we have:

$$\begin{aligned}
&\sum_{j\in I} \overline{\alpha}_j f_j(\widetilde{u}) + \sum_{t\in\mathcal{T}} \overline{\sigma}^\Psi_t \Psi_t(\widetilde{u}) + \sum_{j\in\mathcal{J}} \overline{\sigma}^\theta_j \theta_j(\widetilde{u}) + \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{B}_j \mathcal{B}_j(\widetilde{u}) - \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{A}_j \mathcal{A}_j(\widetilde{u})\\
&- \left(\sum_{j\in I} \overline{\alpha}_j f_j(\hat{a}) + \sum_{t\in\mathcal{T}} \overline{\sigma}^\Psi_t \Psi_t(\hat{a}) + \sum_{j\in\mathcal{J}} \overline{\sigma}^\theta_j \theta_j(\hat{a}) + \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{B}_j \mathcal{B}_j(\hat{a}) - \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{A}_j \mathcal{A}_j(\hat{a})\right) < 0.
\end{aligned} \tag{4.8}$$

In light of the geodesic convexity assumptions on the objective function and constraints, we obtain that the following inequalities are satisfied for every $a \in \mathcal{F}$:

$$
\begin{aligned}
f_j(\widetilde{u}) - f_j(\hat{a}) &\geq \langle \xi_i^f, \exp_{\hat{a}}^{-1}(\widetilde{u})\rangle, && \forall j \in I,\ \xi_i^f \in \partial^c f_j(\hat{a}),\\
\Psi_t(\widetilde{u}) - \Psi_t(\hat{a}) &\geq \langle \xi_t^\Psi, \exp_{\hat{a}}^{-1}(\widetilde{u})\rangle, && \forall t \in L^+(\hat{a}),\ \xi_t^\Psi \in \partial^c \Psi_t(\hat{a}),\\
\theta_j(\widetilde{u}) - \theta_j(\hat{a}) &\geq \langle \xi_j^\theta, \exp_{\hat{a}}^{-1}(\widetilde{u})\rangle, && \forall j \in \mathcal{J}^+(\hat{a}),\ \xi_j^\theta \in \partial^c \theta_j(\hat{a}),\\
-\theta_j(\widetilde{u}) + \theta_j(\hat{a}) &\geq \langle -\xi_j^\theta, \exp_{\hat{a}}^{-1}(\widetilde{u})\rangle, && \forall j \in \mathcal{J}^-(\hat{a}),\ \xi_j^\theta \in \partial^c \theta_j(\hat{a}),\\
\mathcal{A}_j(\widetilde{u}) - \mathcal{A}_j(\hat{a}) &\geq \left\langle \xi_j^{\mathcal{A}}, \exp_{\hat{a}}^{-1}(\widetilde{u})\right\rangle, && \forall j \in \hat{\mathcal{R}}_{0+}^-(\hat{a}),\ \xi_j^{\mathcal{A}} \in \partial^c \mathcal{A}_j(\hat{a}),\\
\mathcal{B}_j(\widetilde{u}) - \mathcal{B}_j(\hat{a}) &\geq \left\langle \xi_j^{\mathcal{B}}, \exp_{\hat{a}}^{-1}(\widetilde{u})\right\rangle, && \forall j \in \mathcal{R}_{+0}^+(\hat{a}) \cup \mathcal{R}_{00}^+,\ \xi_j^{\mathcal{B}} \in \partial^c \mathcal{B}_j(\hat{a}),\\
-\mathcal{A}_j(\widetilde{u}) + \mathcal{A}_j(\hat{a}) &\geq -\left\langle \xi_j^{\mathcal{A}}, \exp_{\hat{a}}^{-1}(\widetilde{u})\right\rangle, && \forall j \in \hat{\mathcal{R}}_{0+}^+(\hat{a}) \cup \hat{\mathcal{R}}_{00}^+(\hat{a}) \cup \hat{\mathcal{R}}_{0-}^+(\hat{a}),\\
& && \xi_j^{\mathcal{A}} \in \partial^c \mathcal{A}_j(\hat{a}).
\end{aligned} \tag{4.9}
$$

As a result, it follows that:

$$
\begin{aligned}
&\sum_{i\in I} \bar{\alpha}_i f_i(\widetilde{u}) + \sum_{t\in\mathcal{T}} \bar{\sigma}_t^\Psi \Psi_t(\widetilde{u}) + \sum_{i\in\mathcal{J}} \bar{\sigma}_i^\theta \theta_i(\widetilde{u}) - \sum_{i\in\mathcal{S}} \bar{\sigma}_i^H \mathcal{A}_i(\widetilde{u}) + \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{B}} \mathcal{B}_i(\widetilde{u})\\
&-\Big(\sum_{i\in I} \bar{\alpha}_i f_i(\hat{a}) + \sum_{t\in\mathcal{T}} \bar{\sigma}_t^\Psi \Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \bar{\sigma}_i^\theta \theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \bar{\sigma}_i^H \mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{B}} \mathcal{B}_i(\hat{a})\Big)\\
&\geq \big\langle \sum_{i\in I} \bar{\alpha}_i \xi_i^f + \sum_{t\in\mathcal{T}} \bar{\sigma}_t^\Psi \xi_t^\Psi + \sum_{i\in\mathcal{J}} \bar{\sigma}_i^\theta \xi_i^\theta - \sum_{i\in\mathcal{S}} \bar{\sigma}_i^H \xi_i^{\mathcal{A}} + \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{B}} \xi_i^{\mathcal{B}},\ \exp_{\hat{a}}^{-1}(\widetilde{u})\big\rangle.
\end{aligned}
$$

In view of the above inequality and (4.8), we arrive at a contradiction. Therefore, we have the following:

$$
f(\hat{a}) = \overline{\mathcal{L}}(\hat{a}, \bar{\sigma}) \in \Phi^W(\bar{\sigma}) = \mathrm{WMin}_{\mathbb{R}_+^m} \left\{\overline{\mathcal{L}}(a, \bar{\sigma}) \mid a \in \mathcal{F}\right\} \tag{4.10}
$$

In view of Theorem 4.4, we infer that $f(\hat{a})$ is a weakly Pareto efficient solution of DV($\hat{a}$). Thus, the proof is complete. □

Remark 4.7. Theorem 4.6 extends Proposition 3.3 established by Tung et al. [21] from the setting of Euclidean space to the framework of a more general space, namely, Hadamard manifolds.

In the following definition, we introduce the notion of weak vector saddle point for the weak vector Lagrangian function of NSIMOPVC in the Hadamard manifold framework.

Definition 4.8. Let us suppose that $\hat{a} \in \mathcal{F}$ and $\bar{\sigma} \in \mathcal{F}_1(\hat{a})$. Then, $(\hat{a}, \bar{\sigma})$ is referred to as a weak saddle point for the function $\overline{\mathcal{L}}$, provided that:

$$
\overline{\mathcal{L}}(a, \bar{\sigma}) \not< \overline{\mathcal{L}}(\hat{a}, \bar{\sigma}) \not< \overline{\mathcal{L}}(\hat{a}, \sigma), \quad \forall a \in \mathcal{F},\ \forall \sigma \in \mathcal{F}_1(\hat{a}).
$$

Remark 4.9. Definition 4.8 extends Definition 3.9 presented by Tung et al. [21] from Euclidean space framework to the setting of Hadamard manifolds, and moreover, generalizes it for a more general class of optimization problems.

Theorem 4.10. *Let us assume that every hypothesis stated in Theorem 4.6 is satisfied. Further, let us assume that $\hat{a}$ is a weakly efficient solution of NSIMOPVC. Then, there exists some $\bar{\sigma} \in \mathcal{F}_1(\hat{a})$ such that $(\hat{a}, \bar{\sigma})$ is a weak saddle point of $\overline{\mathcal{L}}$.*
(ii) If $(\hat{a}, \bar{\sigma}) \in \mathcal{F} \times \mathcal{F}_1(\hat{a})$ is a weak saddle point of $\overline{\mathcal{L}}$, then $f(\hat{a}) \in \Phi^W(\bar{\sigma})$, where $\hat{a}$ and $f(\hat{a})$ are weakly efficient solutions to the primal (P) and the dual $(WVD_L(\hat{a}))$, resp.

Proof. (i) In light of the conclusions of Theorem 4.6, we infer that there exists some $\bar{\sigma} \in \mathcal{F}_1(\hat{a})$, such that

$$
f(\hat{a}) = \overline{\mathcal{L}}(\hat{a}, \bar{\sigma}). \tag{4.11}
$$

Moreover, the following inequality holds true:

$$\overline{\mathcal{L}}(a,\bar{\sigma}) \nprec \overline{\mathcal{L}}(\hat{a},\bar{\sigma}), \quad \forall a \in \mathcal{F}.$$

It suffices to prove that the following holds true:

$$\overline{\mathcal{L}}(\hat{a},\bar{\sigma}) \nprec \overline{\mathcal{L}}(\hat{a},\sigma), \quad \forall \sigma \in \mathcal{F}_1(\hat{a}).$$

By *reductio ad absurdum*, we assume that there exists some $\sigma \in \mathcal{F}_1(\hat{a})$, such that

$$\overline{\mathcal{L}}(\hat{a},\bar{\sigma}) \prec \overline{\mathcal{L}}(\hat{a},\sigma).$$

In light of (4.11), it follows that:

$$f(\hat{a}) \prec f(\hat{a}) + \left(\sum_{t\in\mathcal{T}} \sigma_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \sigma_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{B}}\mathcal{B}_i(\hat{a})\right) e.$$

Hence, for every $\forall i \in I$, we have the following:

$$f_i(\hat{a}) < f_i(\hat{a}) + \left(\sum_{t\in\mathcal{T}} \sigma_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \sigma_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{B}}\mathcal{B}_i(\hat{a})\right).$$

Evidently, we have the following:

$$\sum_{t\in\mathcal{T}} \sigma_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \sigma_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{B}}\mathcal{B}_i(\hat{a}) > 0. \tag{4.12}$$

However, from the feasibility conditions, it readily follows that:

$$\sum_{t\in\mathcal{T}} \sigma_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \sigma_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{B}}\mathcal{B}_i(\hat{a}) \leq 0.$$

Evidently, this a contradiction to (4.12). Thus we have:

$$\overline{\mathcal{L}}(\hat{a},\bar{\sigma}) \nprec \overline{\mathcal{L}}(\hat{a},\sigma), \quad \forall \sigma \in \mathcal{F}_1(\hat{a}),$$

This completes the proof.

(ii) From the given hypothesis, we have that $(\hat{a},\bar{\sigma}) \in \mathcal{F} \times \mathcal{F}_1(\hat{a})$ be a weak saddle point of $\overline{\mathcal{L}}$. Consequently, the following holds:

$$\overline{\mathcal{L}}(\hat{a},\bar{\sigma}) \nprec \overline{\mathcal{L}}(\hat{a},\sigma), \quad \forall \sigma \in \mathcal{F}_1(\hat{a}).$$

Equivalently, we have the following:

$$\begin{aligned} f(\hat{a}) &+ \left(\sum_{t\in\mathcal{T}} \bar{\sigma}_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \bar{\sigma}_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{B}}\mathcal{B}_i(\hat{a})\right) e \\ \nprec f(\hat{a}) &+ \left(\sum_{t\in\mathcal{T}} \sigma_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \sigma_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{B}}\mathcal{B}_i(\hat{a})\right) e. \end{aligned}$$

By putting $\sigma = 0$ in the above expression, we arrive at the following:

$$f(\hat{a}) + \left(\sum_{t\in\mathcal{T}} \bar{\sigma}_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \bar{\sigma}_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{B}}\mathcal{B}_i(\hat{a})\right) e \nprec f(\hat{a}).$$

Continuing along similar lines as in the proof of Theorem 4.2, we have:

$$\sum_{t\in\mathcal{T}} \bar{\sigma}_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \bar{\sigma}_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{B}}\mathcal{B}_i(\hat{a}) \leq 0.$$

Let us now consider the case:

$$\sum_{t\in\mathcal{T}}\bar{\sigma}_t^{\Psi}\Psi_t(\hat{a})+\sum_{i\in\mathcal{J}}\bar{\sigma}_i^{\theta}\theta_i(\hat{a})-\sum_{i\in\mathcal{S}}\bar{\sigma}_i^{\mathcal{A}}\mathcal{A}_i(\hat{a})+\sum_{i\in\mathcal{S}}\bar{\sigma}_i^{\mathcal{B}}\mathcal{B}_i(\hat{a})<0.$$

In such a scenario, we have:

$$f(\hat{a})+\left(\sum_{t\in\mathcal{T}}\bar{\sigma}_t^{\Psi}\Psi_t(\hat{a})+\sum_{i\in\mathcal{J}}\bar{\sigma}_i^{\theta}\theta_i(\hat{a})-\sum_{i\in\mathcal{S}}\bar{\sigma}_i^{\mathcal{A}}\mathcal{A}_i(\hat{a})+\sum_{i\in\mathcal{S}}\bar{\sigma}_i^{\mathcal{B}}\mathcal{B}_i(\hat{a})\right)e\prec f(\hat{a}).$$

Clearly, this is a contradiction to 4. As a result, we have:

$$\sum_{t\in\mathcal{T}}\bar{\sigma}_t^{\Psi}\Psi_t(\hat{a})+\sum_{i\in\mathcal{J}}\bar{\sigma}_i^{\theta}\theta_i(\hat{a})-\sum_{i\in\mathcal{S}}\bar{\sigma}_i^{\mathcal{A}}\mathcal{A}_i(\hat{a})+\sum_{i\in\mathcal{S}}\bar{\sigma}_i^{\mathcal{B}}\mathcal{B}_i(\hat{a})=0.$$

Hence, it follows that:

$$\overline{\mathcal{L}}(\hat{a},\bar{\sigma})=f(\hat{a}).$$

On the other hand, since $(\hat{a},\bar{\sigma})$ is a weak saddle point of $\overline{\mathcal{L}}$, we have that:

$$\overline{\mathcal{L}}(a,\bar{\sigma})\not\prec\overline{\mathcal{L}}(\hat{a},\bar{\sigma}),\quad\forall a\in\mathcal{F}$$

Hence, we arrive at the following:

$$f(\hat{a})=\overline{\mathcal{L}}(\hat{a},\bar{\sigma})\in\operatorname*{WMin}_{\mathbb{R}^m_+}\left\{\overline{\mathcal{L}}(a,\bar{\sigma})\mid a\in\mathcal{F}\right\}=\omega^W(\bar{\sigma}).$$

In light of Theorem 4.4, it follows that $f(\hat{a})$ is a weak Pareto efficient solution of DV$(\hat{a})$. Thus the proof is complete. □

Remark 4.11. Theorem 4.10 extends Proposition 3.10 established by Tung et al. [21] from the setting of Euclidean space to the framework of a more general space, namely, Hadamard manifolds.

Theorem 4.12. *Let $\hat{a}\in\mathcal{F}$ be strong stationary element of NSIMOPVC for some $\bar{\alpha}\in\mathbb{R}^m_+$ and $\sum_{i\in I}\bar{\alpha}_i=1$. Moreover, suppose that the following functions:*

$$\begin{aligned}
&f_i\ (i\in I), && \Psi_t\ (t\in L^+(\hat{a})),\\
&\theta_i\ \left(i\in\mathcal{J}^+(\hat{a})\right), && -\theta_i\ (i\in\mathcal{J}^-(\hat{a})),\\
&\mathcal{A}_i\ \left(i\in\hat{\mathcal{R}}^-_{0+}(\hat{a})\right), && -\mathcal{A}_i\ \left(i\in\hat{\mathcal{R}}^+_{0+}(\hat{a})\cup\hat{\mathcal{R}}^+_{00}(\hat{a})\cup\hat{\mathcal{R}}^+_{0-}(\hat{a})\right),\\
&\mathcal{B}_i\ \left(i\in\mathcal{R}^+_{\pm0}(\hat{a})\cup\mathcal{R}^+_{00}(\hat{a})\right), &&
\end{aligned}$$

be geodesic convex at $\hat{a}$. Then there exists $\bar{\sigma}\in\sigma(\hat{a})\times\mathbb{R}^q\times\mathbb{R}^p\times\mathbb{R}^p$, such that $(\hat{a},\bar{\sigma})$ is a weak saddle point of $\overline{\mathcal{L}}$.

Proof. To begin with, we claim that the following inequality is satisfied:

$$\overline{\mathcal{L}}(a,\bar{\sigma})\not\prec\overline{\mathcal{L}}(\hat{a},\bar{\sigma}),\quad\forall a\in\mathcal{F}.$$

By *reductio ad absurdum*, we assume that there exists some $\widetilde{w}\in\mathcal{F}$, such that

$$\overline{\mathcal{L}}(\widetilde{w},\bar{\sigma})\prec\overline{\mathcal{L}}(\hat{a},\bar{\sigma}).$$

As a result, we have:

$$\begin{aligned}
&f(\widetilde{w})+\left(\sum_{t\in\mathcal{T}}\bar{\sigma}_t^{\Psi}\Psi_t(\widetilde{w})+\sum_{i\in\mathcal{J}}\bar{\sigma}_i^{\theta}\theta_i(\widetilde{w})-\sum_{i\in\mathcal{S}}\bar{\sigma}_i^{\mathcal{A}}\mathcal{A}_i(\widetilde{w})+\sum_{i\in\mathcal{S}}\bar{\sigma}_i^{\mathcal{B}}\mathcal{B}_i(\widetilde{w})\right)e\\
\prec\ &f(\hat{a})+\left(\sum_{t\in\mathcal{T}}\bar{\sigma}_t^{\Psi}\Psi_t(\hat{a})+\sum_{i\in\mathcal{J}}\bar{\sigma}_i^{\theta}\theta_i(\hat{a})-\sum_{i\in\mathcal{S}}\bar{\sigma}_i^{\mathcal{A}}\mathcal{A}_i(\hat{a})+\sum_{i\in\mathcal{S}}\bar{\sigma}_i^{\mathcal{B}}\mathcal{B}_i(\hat{a})\right)e.
\end{aligned}$$

Therefore, for every $i \in I$, we have:

$$f_i(\widetilde{w}) + \sum_{t\in\mathcal{T}} \bar{\sigma}_t^{\Psi}\Psi_t(\widetilde{w}) + \sum_{i\in\mathcal{J}} \bar{\sigma}_i^{\theta}\theta_i(\widetilde{w}) - \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{A}}\mathcal{A}_i(\widetilde{w}) + \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{B}}\mathcal{B}_i(\widetilde{w})$$
$$< f_i(\hat{a}) + \sum_{t\in\mathcal{T}} \bar{\sigma}_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \bar{\sigma}_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{B}}\mathcal{B}_i(\hat{a}).$$

We recall that $\bar{\alpha} \in \mathbb{R}^m_+$ and $\sum_{i\in I} \bar{\alpha}_i = 1$. Let us now multiply both sides of the above inequalities with $\bar{\alpha}_i$ and subsequently, add them up. Hence, we have:

$$\sum_{i\in I} \bar{\alpha}_i f_i(\widetilde{w}) + \sum_{t\in\mathcal{T}} \bar{\sigma}_t^{\Psi}\Psi_t(\widetilde{w}) + \sum_{i\in\mathcal{J}} \bar{\sigma}_i^{\theta}\theta_i(\widetilde{w}) - \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{A}}\mathcal{A}_i(\widetilde{w}) + \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{B}}\mathcal{B}_i(\widetilde{w})$$
$$-\left(\sum_{i\in I} \bar{\alpha}_i f_i(\hat{a}) + \sum_{t\in\mathcal{T}} \bar{\sigma}_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \bar{\sigma}_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{B}}\mathcal{B}_i(\hat{a})\right) < 0.$$

Continuing along similar lines as in the proof of Theorem 4.6, we arrive at a contradiction. Hence, we have:

$$\overline{\mathcal{L}}(a, \bar{\sigma}) \not\prec \overline{\mathcal{L}}(\hat{a}, \bar{\sigma}), \quad \forall a \in \mathcal{F}.$$

Now, we claim that the following holds:

$$\overline{\mathcal{L}}(\hat{a}, \bar{\sigma}) \not\prec \overline{\mathcal{L}}(\hat{a}, \sigma), \forall \sigma \in \mathcal{F}_1(\hat{a}).$$

By *reductio ad absurdum*, we assume that there exists $\sigma \in \mathcal{F}_1(\hat{a})$, such that:

$$\overline{\mathcal{L}}(\hat{a}, \bar{\sigma}) \prec \overline{\mathcal{L}}(\hat{a}, \sigma).$$

Then, we have the following:

$$f(\hat{a}) + \left(\sum_{t\in\mathcal{T}} \bar{\sigma}_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \bar{\sigma}_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{B}}\mathcal{B}_i(\hat{a})\right) e$$
$$\prec f(\hat{a}) + \left(\sum_{t\in\mathcal{T}} \sigma_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \sigma_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{B}}\mathcal{B}_i(\hat{a})\right) e.$$

Moreover, $\hat{a}$ is a strong stationary element of NSIMOPVC. Hence, we have:

$$\sum_{t\in\mathcal{T}} \bar{\sigma}_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \bar{\sigma}_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \bar{\sigma}_i^{\mathcal{B}}\mathcal{B}_i(\hat{a}) = 0.$$

Evidently, one has:

$$\sum_{t\in\mathcal{T}} \sigma_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \sigma_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{B}}\mathcal{B}_i(\hat{a}) > 0.$$

In view of the facts that $\hat{a} \in \mathcal{F}$ and $\sigma \in \mathcal{F}_1(\hat{a})$, we get:

$$\sum_{t\in\mathcal{T}} \sigma_t^{\Psi}\Psi_t(\hat{a}) + \sum_{i\in\mathcal{J}} \sigma_i^{\theta}\theta_i(\hat{a}) - \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{A}}\mathcal{A}_i(\hat{a}) + \sum_{i\in\mathcal{S}} \sigma_i^{\mathcal{B}}\mathcal{B}_i(\hat{a}) \leq 0.$$

which is a contradiction to (4). Hence, it is established that $(\hat{a}, \bar{\sigma})$ is a weak saddle point of $\overline{\mathcal{L}}$. This completes the proof.

□

Remark 4.13. Theorem 4.10 extends Proposition 3.11 established by Tung et al. [21] from the setting of Euclidean space to the framework of a more general space, namely, Hadamard manifolds.

Example 4.14. Consider the Hadamard manifold $\mathscr{M} \subset \mathbb{R}^2$ defined by

$$\mathscr{M} := \{z = (z_1, z_2) \in \mathbb{R}^2 : \ z_1, z_2 > 0\}.$$

Then $\mathscr{M}$ is a Hadamard manifold (see, [24]). At any $\hat{z} \in \mathscr{M}$, we have

$$T_{\hat{z}}\mathscr{M} = \mathbb{R}^2.$$

The corresponding metric on $\mathscr{M}$ is given by

$$\langle w_1, w_2\rangle_{\hat{z}} = \langle \mathscr{G}(\hat{z})w_1, w_2\rangle,$$

$w_1, w_2 \in T_{\hat{z}}\mathscr{M}$, where,

$$\mathscr{G}(\hat{z}) = \operatorname{diag}(\frac{1}{\hat{z}_1^2}, \frac{1}{\hat{z}_2^2}).$$

The inverse of the exponential function $\exp_{\hat{z}} : T_{\hat{z}}\mathscr{M} \to \mathscr{M}$ for any $v \in T_{\hat{z}}\mathscr{M}$ is given by

$$\exp_{\hat{z}}(v) = (\hat{z}_1 e^{\frac{v_1}{\hat{z}_1}}, \hat{z}_2 e^{\frac{v_2}{\hat{z}_2}}), \ \forall v = (v_1, v_2) \in \mathscr{M}.$$

Consider the following problem $(P2)$ which is an (NSIMOPVC):

$$\begin{aligned} (P2) \quad \text{Minimize } \Phi(y) &= (\Phi_1(y), \Phi_2(y)) := (|y_1|, \log y_2), \\ \text{subject to } \ \Psi_t(y) &:= t(2 - \ln y_1 - \ln y_2) \leq 0, \quad t \in \mathbb{N}, \\ \mathcal{M}(y) &:= \ln y_1 - 1 \geq 0, \\ \mathcal{N}(y)^T\mathcal{M}(y) &:= (e - y_2)(\ln y_1 - 1) \leq 0, \end{aligned}$$

where, $\Phi_i : \mathscr{M} \to \mathbb{R}\ (i = 1, 2)$, $\Psi_t : \mathscr{M} \to \mathbb{R}\ (t \in \mathbb{N})$, $\mathcal{M} : \mathscr{M} \to \mathbb{R}$ and $\mathcal{N} : \mathscr{M} \to \mathbb{R}$. The feasible set F for the problem is

$$: F = \{y \in \mathscr{M} : y_1 \geq e, \ y_2 \geq e\}.$$

Let $y^* = (e, e) \in F$. We have the following:

$$\partial^c\Phi_1(y) = \begin{cases} \{(y_1^2, 0)^T\}, & \text{if } y_1 > 0, \\ \operatorname{co}\{(y_1^2, 0)^T, (-y_1^2, 0)^T\}, & \text{if } y_1 = 0, \\ \{(-y_1^2, 0)^T\}, & \text{if } y_1 < 0, \end{cases} \qquad \partial^c\Psi_t(y) = \left\{ \begin{pmatrix} -ty_1 \\ -ty_2 \end{pmatrix} \right\}, \quad \partial^c\mathcal{M}(y) = \left\{ \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \right\},$$

$$\partial^c\Phi_2(y) = \left\{ \begin{pmatrix} 0 \\ y_2 \end{pmatrix} \right\}, \qquad \partial^c\mathcal{N}(y) = \left\{ \begin{pmatrix} 0 \\ -y_2^2 \end{pmatrix} \right\}.$$

We observe that (NMSIPVC-ACQ) holds at y^*. Moreover, we can verify the fact that y^* is a weak Pareto efficient solution of (P2). Let us choose multipliers as $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$, $\sigma_t^\Psi = 0\ (t \in \mathbb{N})$, $\sigma^{\mathcal{M}} = \frac{e}{2}$, $\sigma^{\mathcal{N}} = \frac{1}{2e}$. Then for $\xi_1^\Phi = (e^2, 0)^T \in \partial^c\Phi_1(y^*)$, $\xi_2^\Phi = (0, e)^T \in \partial^c\Phi_2(y^*)$, $\xi_t^\Psi = (-te, -te)^T \in \partial^c\Psi_t(y^*)$ $(t \in \mathbb{N})$, $\xi^{\mathcal{M}} = (e, 0)^T \in \partial^c\mathcal{M}(y^*)$, $\xi^{\mathcal{N}} = (0, -e^2)^T \in \partial^c\mathcal{N}(y^*)$, we have the following:

$$\alpha_1\xi_1^\Phi + \alpha_2\xi_2^\Phi + \sum_{t\in\mathbb{N}} \sigma_t^\Psi\xi_t^\Psi - \sigma^{\mathcal{M}}\xi^{\mathcal{M}} + \sigma^{\mathcal{N}}\xi^{\mathcal{N}} = (0, 0)^T.$$

It can be verified that the objective function and constraints are geodesic convex at y^*. Hence all the assumptions and conclusions of Theorem 5 holds true. Thus (y^*, σ) is a saddle point for the corresponding Lagrangian function.

5. Conclusions

In this article, we have studied a class of NSIMOPVC on Hadamard manifolds. We have introduced the scalarized Lagrange-type dual problem and the vector Lagrange-type dual problem related to NSIMOPVC, in the framework of Hadamard manifolds. We have derived weak and strong duality theorems relating NSIMOPVC and the corresponding dual problems under suitable geodesic convexity assumptions. Moreover, we have established saddle point optimality conditions for NSIMOPVC in the setting of Hadamard manifolds. Several non-trivial numerical examples have been provided to demonstrate the significance of the derived results.

The results derived in this paper generalize, extend and unify several notable results existing in the literature. For instance, the results derived in the paper extend the corresponding results derived by Tung et al. [21] from Euclidean space setting to the framework of Hadamard manifold, as well as generalize them for a more general class of optimization problems.

The results presented in this article unfold several avenues for future research. For instance, it would be an interesting research problem to derive saddle point optimality criteria and duality results for NSIMOPVC by employing Mordukhovich limiting subdifferentials on Riemannian manifolds.

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References

[1] W. Achtziger and C. Kanzow. Mathematical programs with vanishing constraints: optimality conditions and constraint qualifications. *Mathematical Programming*, 114:69–99, 2008.

[2] A. Barani. Generalized monotonicity and convexity for locally Lipschitz functions on Hadamard manifolds. *Differential Geometry-Dynamical Systems*, 15:26-37, 2013.

[3] A. Charnes, W. W. Cooper, and K. O. Kortanek. Duality, Haar programs, and finite sequence spaces. *Proceedings of the National Academy of Sciences of the United States of America,* 48:783-786, 1962.

[4] T. Chuong and D. Kim. Nonsmooth semi-infinite multiobjective optimization problems. *Journal of Optimization Theory and Applications,* 160:748-762, 2014.

[5] D. Dorsch, V. Shikhman, and O. Stein. Mathematical programs with vanishing constraints: critical point theory. *Journal of Global Optimization,* 52(3):591–605, 2012.

[6] H. Djelassi, A. Mitsos, and O. Stein. Recent advances in nonconvex semi-infinite programming: Applications and algorithms. *EURO Journal on Computational Optimization,* 9:Article ID 100006, 2021.

[7] P. T. Fletcher, J. Moeller, J. M. Phillips, and S. Venkatasubramanian. Horoball hulls and extents in positive definite space. In F. Dehne, J. Iacono, J. R. Sack, editors, *Algorithms and Data Structures, Lecture Notes in Computer Science*, vol 6844, pages 386–398, WADS 2011, Springer, Berlin, Heidelberg.

[8] M. Goberna and M. Lopez. Linear semi-infinite programming theory: An updated survey. *European Journal of Operational Research*, 143:390-405, 2002.

[9] S. M. Guu, Y. Singh, and S. K. Mishra. On strong KKT type sufficient optimality conditions for multiobjective semi-infinite programming problems with vanishing constraints. *Journal of Inequalities and Applications,* 2017:Article ID 282, 2017.

[10] A. Haar. Über lineare Ungleichungen. *Acta Mathematica Szeged*, 2:1-14, 1924.

[11] R. Hettich and K. O. Kortanek. Semi-infinite programming: theory, methods, and applications. *SIAM Review*, 35(3):380-429, 1993.

[12] T. Hoheisel and C. Kanzow. First- and second-order optimality conditions for mathematical programs with vanishing constraints. *Applications of Mathematics*, **52**:495–514, 2007.

[13] T. Hoheisel and C. Kanzow. Stationary conditions for mathematical programs with vanishing constraints using weak constraint qualifications. *Journal of Mathematical Analysis and Applications*, 337(1):292–310, 2008.

[14] T. Hoheisel and C. Kanzow. On the Abadie and Guignard constraint qualifications for mathematical programmes with vanishing constraints. *Optimization,* 58(4):431–448, 2009.

[15] F. John. Extremum problems with inequalities as subsidiary conditions. Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, Interscience Publishers Inc., Pages 187–204, New York, 1948.

[16] M. M. Karkhaneei and N. Mahdavi-Amiri. Nonconvex weak sharp minima on Riemannian manifolds. *Journal of Optimization Theory and Applications,* 183:85–104, 2019.

[17] C. Kanzow and A. Schwartz. Mathematical programs with equilibrium constraints: enhanced Fritz John-conditions, new constraint qualifications, and improved exact penalty results. *SIAM Journal on Optimization,* 20:2730-2753, 2010.

[18] A. Kristály, C. Li, G. Lopez-Acedo, and A. Nicolae. What do 'convexities' imply on Hadamard manifolds? *Journal of Optimization Theory and Applications,* 170:1068–1074, 2016.

[19] M. López and G. Still. Semi-infinite programming. *European Journal of Operational Research,* 180(2):491-518, 2007.

[20] S. K. Mishra, V. Singh, V. Laha, and R. N. Mohapatra. On constraint qualifications for multiobjective optimization problems with vanishing constraints. In H. Xu, S. Wang, SY. Wu, editors, *Optimization Methods, Theory and Applications*, pages 95–135. Springer, Heidelberg, 2015.

[21] L. T. Tung, D. H. Tam, and T. T. Khai. Lagrange duality and saddle point optimality conditions for multiobjective semi-infinite programming with vanishing constraints. *Numerical Functional Analysis and Optimization,* 45:44–81, 2024.

[22] C. Udrişte. *Convex Functions and Optimization Methods on Riemannian Manifolds.* Springer, Berlin, 2013.

[23] B. B. Upadhyay and A. Ghosh. On constraint qualifications for mathematical programming problems with vanishing constraints on Hadamard manifolds. *Journal of Optimization Theory and Applications,* 199:1–35, 2023.

[24] B. B. Upadhyay, A. Ghosh, N. Kanzi, and H. Soroush. Constraint qualifications for nonsmooth multiobjective programming Problems with switching constraints on Hadamard manifolds. *Bulletin of the Malaysian Mathematical Sciences Society,* 47:Article ID 103, 2024.

[25] B. B. Upadhyay, A. Ghosh, P. Mishra, and S. Treanţă. Optimality conditions and duality for multiobjective semi-infinite programming problems on Hadamard manifolds using generalized geodesic convexity. *RAIRO - Operations Research,* 56(4):2037–2065, 2022.

[26] J. Wang, X. Wang, C. Li, and J. C. Yao. Convergence analysis of gradient algorithms on Riemannian manifolds without curvature constraints and application to Riemannian mass. *SIAM Journal on Optimization,* 31:172–199, 2021.