FIXED POINT METHODS AND OPTIMIZATION
Volume 3 (2026), No. 1, 60–75
https://doi.org/10.69829/fpmo-026-0301-ta05
Tulipa Opera Scholarum

A DOUBLE INERTIAL EXTRAGRADIENT ALGORITHM WITH SELF-ADAPTIVE STEPSIZES FOR SOLVING VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS

XIAN-JUN LONG[1], SI-JIE YU[1], ZAI-YUN PENG[2,*], YEKINI SHEHU[3], AND SIMEON REICH[4]

[1] *School of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, P.R.China*
[2] *School of Mathematics, Yunnan Normal University, Kunming, 650092, P.R. China and Yunnan Key Laboratory of Modern Analytical Mathematics and Applications, Kunming, 650500, P.R. China*
[3] *School of Mathematical Sciences, Zhejiang Normal University, Jinhua, 321004, P.R. China*
[4] *Department of Mathematics, The Technion-Israel Institute of Technology, 3200003 Haifa, Israel*

Abstract. In this paper, we present a double inertial extragradient algorithm with self-adaptive stepsizes for finding a common solution of a pseudomonotone variational inequality and a fixed point problem with a quasi-nonexpansive mapping in Hilbert spaces. The self-adaptive stepsize rule allows the stepsizes to increase and converge, which may accelerate the convergence of the algorithm. We establish strong convergence theorems under some modern conditions. Some numerical experiments illustrate the performances and advantages of our proposed algorithm.

Keywords. Variational inequality, Fixed point, Pseudomonotone mapping, Inertial method, Self-adaptive stepsize, Strong convergence.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let $C \subset H$ be a nonempty closed convex subset. Let $A : H \to H$ be a nonlinear operator. The variational inequality problem is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0,\ \forall x \in C. \tag{1.1}$$

Let $\mathrm{VI}(C, A)$ denote the solution set of problem (1.1).

Variational inequalities have received much attention due to their applications including economics, transportation, nonlinear equations, and optimal control problems; see, for example [1, 7, 17, 18]. Several iterative algorithms have been presented to solve problem (1.1); see, for example [2, 3, 4, 11, 13, 16, 21, 23, 24, 25, 26, 29, 31, 33, 34, 35, 43, 45].

Another important tropic is fixed point problems. Let $T : H \to H$ be a nonlinear operator. A point $x \in H$ is called a fixed point of T if $T(x) = x$. Denoted by $Fix(T)$ the set of fixed points of T. The purpose of this paper is finding a point $z \in C$ such that:

$$z \in \mathrm{VI}(C, A) \cap Fix(T). \tag{1.2}$$

There are many numerical algorithms have been proposed for finding a solution of problem (1.2); see, for example [5, 8, 15, 20, 28, 30, 32, 36, 38, 39, 40, 42, 46].

[*]Corresponding author.
E-mail address: xianjunlong@ctbu.edu.cn (X. J. Long), sijyu387@163.com (S. J. Yu), pengzaiyun@126.com (Z. Y. Peng), yekini.shehu@zjnu.edu.cn (Y. Shehu), and sreich@technion.ac.il (S. Reich)
2020 Mathematics Subject Classification: 49J40, 65K15, 47H09, 90C33
Accepted: December 10, 2025.

It is well known that the inertial-type algorithm has been investigated because it can speed up the convergence of the algorithm; see, "for example". Singh and Chandok et al. [36] proposed the following Mann-type inertial extragradient algorithms:

$$\begin{cases} w_n = x_n + \alpha(x_n - x_{n-1}), \\ u_n = x_n + \beta_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = (1-\gamma_n)w_n + \gamma_n T z_n, \\ z_n = y_n - \lambda_n(Ay_n - Au_n), \\ \lambda_{n+1} = \begin{cases} \min\{\frac{\mu\|u_n - y_n\|}{\|A(u_n)-A(y_n)\|}, \lambda_n + \rho_n\}, & \text{if } A(y_n) - A(y_n) \neq 0, \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases} \end{cases} \tag{1.3}$$

They proved that the sequence $\{x_n\}$ generated by (1.3) converges weakly to an element of $\mathrm{VI}(C, A) \cap Fix(T)$ when A is pseudomonotone and L-Lipschitz continuous, and T is quasinonexpansive.

Recently, Li and Xie [19] proposed a double inertial subgradient extragradient algorithm with self-adaptive stepsize for solving problem (1.2) as follows:

$$\begin{cases} w_n = x_n + \alpha(x_n - x_{n-1}), \\ u_n = x_n + \beta(x_n - x_{n-1}), \\ y_n = P_C(u_n - \lambda_n A(u_n)), \\ t_n = P_{T_n}(u_n - \tau\lambda_n A(y_n)), \\ T_n := \{x \in H : \langle u_n - \lambda_n F(u_n) - y_n, x - y_n\rangle \leq 0\}, \\ x_{n+1} = (1-\gamma_n)w_n + \gamma_n T(t_n), \\ \lambda_{n+1} = \begin{cases} \min\{\mu\frac{\|u_n - y_n\|^2 + \|t_n - y_n\|^2}{2\langle A(u_n)-A(y_n), t_n - y_n\rangle}, \lambda_n + \rho_n\}, & \text{if } \langle A(u_n) - A(y_n), t_n - y_n\rangle > 0, \\ \lambda_n + \rho_n, & \text{otherwise,} \end{cases} \end{cases} \tag{1.4}$$

where A is pseudomonotone and L-Lipschitz continuous, and T is quasinonexpansive. They proved that the sequence $\{x_n\}$ generated by (1.4) converges weakly to an element of $\mathrm{VI}(C, A) \cap Fix(T)$. In addition, they also derived a strong convergence theorem by replacing pseudomontone with strongly pseudomonotone mappings.

Motivated and inspired by the above works, we present a double inertial Tseng extragradient algorithm with self-adaptive step size for solving problem (1.2). Under some suitable conditions, we prove a strong convergence theorem for the proposed algorithm. Numerical experiments illustrate the performances and advantages of the proposed algorithm.

2. Preliminaries

Let H be a real Hilbert space and $C \subset H$ be a nonempty closed convex subset. The weak convergence and strong convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively, as $n \to \infty$. Let $\mathbb{N}$ denote the set of positive integers.

The metric projection $P_C : H \to C$ is defined by $P_C(x) = \arg\min\{\|z - x\| : z \in C\},\ \forall x \in H$. It is easy to see that P_C is nonexpansive.

Lemma 2.1. [9] *Let $x \in H$ and $z \in C$. Then*

$$z = P_C(x) \Leftrightarrow \langle x - z, z - y\rangle \geq 0, \forall y \in C.$$

Lemma 2.2. *For each $x, y \in H$ and $\alpha \in \mathbb{R}$, we have*

$$\begin{aligned} \|x+y\|^2 &\leq \|x\|^2 + 2\langle y, x+y\rangle, \\ \|x+y\|^2 &= \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \\ \|\alpha x + (1-\alpha)y\|^2 &= \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \end{aligned}$$

Definition 2.3. Let $F : H \to H$ be a mapping. F is said to be L-Lipschitz continuous with $L > 0$ if

$$\|F(x) - F(y)\| \leq L\|x - y\|, \ \forall x, y \in H.$$

If $L = 1$, then F is called a nonexpansive mapping. If $L \in (0, 1)$, then F is called a contractive mapping.

Definition 2.4. Let $F : H \to H$ be a mapping with $Fix(F) \neq \emptyset$.

(a) F is said to be quasi-nonexpansive if

$$\|F(x) - p\| \leq \|x - p\|, \ \forall x \in H, \ \forall p \in Fix(F).$$

(b) $I - F$ is said to be demiclosed at zero if for any $\{x_n\}$ in H, the following implication holds:

$$x_n \rightharpoonup x \quad \text{and} \quad (I - F)(x_n) \to 0 \Rightarrow x \in Fix(T).$$

Definition 2.5. A mapping $F : H \to H$ is said to be

(a) monotone if $\langle F(x) - F(y), x - y\rangle \geq 0, \ \forall x, y \in H$.
(b) pseudomonotone if $\langle F(x), y - x\rangle \geq 0 \Rightarrow \langle F(y), y - x\rangle \geq 0, \ \forall x, y \in H$.

Remark 2.6. Clearly, $(a) \Rightarrow (b)$. But the converse is not true in general.

Lemma 2.7. [44] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \beta_n)a_n + \beta_n b_n, \forall n \geq 0,$$

where $\{\beta_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that: (i) $\sum_{n=0}^{\infty} \beta_n = \infty$; (ii) $\limsup_{n\to\infty} b_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8. [22] *Let $\{a_n\}$ be a sequence of non-negative real numbers such that there exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $a_{n_i} \leq a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \textit{and} \quad a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest integer n in the set $\{1, 2, \ldots, k\}$ such that $a_n < a_{n+1}$.

Lemma 2.9. [6] *Let $A : C \to H$ be pseudomonotone and continuous. Then $x^* \in \mathrm{VI}(C, A)$ if and only if*

$$\langle Ax, x - x^*\rangle \geq 0, \ \forall x \in C.$$

We consider the following assumptions:
(C1) $\mathrm{VI}(C, A) \cap Fix(T) \neq \emptyset$.
(C2) $A : H \to H$ is L-Lipschizt continuous on H, where L may be unknown.
(C3) $A : H \to H$ is pseudomonotone and satisfies the following property:

$$\{x_n\} \subset C, \ x_n \rightharpoonup z \Rightarrow \|Az\| \leq \liminf_{n\to\infty} \|Ax_n\|. \tag{2.1}$$

(C4) $f : H \to H$ is δ-contraction with $\delta \in [0, 1)$, and $T : H \to H$ is a quasi-nonexpansive mapping such that $I - T$ is demiclosed at zero.

3. Main Results

In this section, we present a new adaptive algorithm with double inertial steps for solving problem (1.2) and analyze the convergence of the proposed algorithm.

The algorithm proposed in this paper is as follows:

Algorithm 1:

Step 1. Given $\lambda_0 > 0$, $0 < \eta_1 < \eta_0 < \sigma < 1$, $\{\xi_n\}$ is a sequence of positive real numbers with $\sum_{n=0}^{+\infty} \xi_n < +\infty$. Let $\alpha_n \in [0,1]$, $\beta_n \in [0,1]$, $\gamma_n \in [0,1]$ and $\theta_n \in [0,1]$. Let $x_0, x_1 \in H$ be arbitrary.

Step 2. Compute

$$w_n = x_n + \alpha_n(x_n - x_{n-1}),$$
$$u_n = x_n + \beta_n(x_n - x_{n-1}).$$

Step 3. Compute

$$y_n = P_C(u_n - \lambda_n A u_n),$$

Step 4. Compute

$$x_{n+1} = \theta_n f(w_n) + (1-\theta_n)[\gamma_n T z_n + (1-\gamma_n) z_n],$$

where $z_n = y_n - \lambda_n(Ay_n - Au_n)$.

Step 5. Update λ_{n+1}.

$$\lambda_{n+1} := \frac{\eta_1 \|y_n - u_n\|}{\|Ay_n - Au_n\|}, \text{ if } \|Ay_n - Au_n\| > \frac{\eta_0}{\lambda_n}\|y_n - u_n\|; \text{ else } \lambda_{n+1} := (1+\xi_n)\lambda_n. \tag{3.1}$$

Set $n := n+1$ and return to Step 1.

Remark 3.1. (a) We use a double inertial technique and a self-adaptive stepsize rule which allows the stepsize to increase and converge in Algorithm 1, which will accelerate the convergence of our algorithm.

(b) Condition (2.1) is strictly weaker than the sequentially weakly continuous assumption used in [10, 32]

(c) The information of the Lipschitz constant of A is not necessary to be known in Algorithm 1.

Lemma 3.2. [12, Lemma 3.2] *Let $A : H \to H$ be L-Lipschitz continuous and $\{\lambda_n\}$ be generated by (3.1) in Algorithm 3.1. Then*

(i) $\lambda_n \geq \lambda^* := \min\{\frac{\eta_1}{L}, \lambda_0\}, \ \forall n \geq 1$;

(ii) $\{\lambda_n\}$ *is convergent;*

(iii) there exists $N \in \mathbb{N}$ *such that* $\lambda_{n+1} \geq \lambda_n, \forall n \geq N$.

Lemma 3.3. *Assume that (C1)-(C3) hold. Let $\{u_n\}$ and $\{y_n\}$ be the sequences generated by Algorithm 3.1. If there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightharpoonup z \in H$ and $\lim_{k\to\infty} \|u_{n_k} - y_{n_k}\| = 0$, then $z \in \mathrm{VI}(C, A)$.*

Proof. Let $u_{n_k} \rightharpoonup z$ and $\lim_{k\to\infty} \|u_{n_k} - y_{n_k}\| = 0$. Since $\{y_n\} \subset C$ and C is a closed set, $z \in C$. By the definition of y_{n_k},

$$\langle u_{n_k} - \lambda_{n_k} A u_{n_k} - y_{n_k}, x - y_{n_k} \rangle \leq 0, \forall x \in C,$$

or equivalently

$$\frac{1}{\lambda_{n_k}} \langle u_{n_k} - y_{n_k}, x - y_{n_k} \rangle + \langle A u_{n_k}, y_{n_k} - u_{n_k} \rangle \leq \langle A u_{n_k}, x - u_{n_k} \rangle, \forall x \in C. \tag{3.2}$$

From Lemma 3.1, we have $\liminf_{k\to\infty} \lambda_{n_k} > 0$. It follows from $u_{n_k} \rightharpoonup z$ that $\{u_{n_k}\}$ is bounded. Since A is Lipschitz continuous on H, $\{Au_{n_k}\}$ is bounded. Note that $\lim_{k\to\infty} \|u_{n_k} - y_{n_k}\| = 0$, we have $\{y_{n_k}\}$ is also bounded. In (3.2), let $k \to \infty$, we have

$$\liminf_{k\to\infty} \langle A u_{n_k}, x - u_{n_k} \rangle \geq 0. \tag{3.3}$$

Observe that

$$\langle Ay_{n_k}, x - y_{n_k}\rangle = \langle Ay_{n_k} - Au_{n_k}, x - u_{n_k}\rangle + \langle Au_{n_k}, x - u_{n_k}\rangle + \langle Ay_{n_k}, u_{n_k} - y_{n_k}\rangle. \tag{3.4}$$

According to the Lipschitz continuity of A and $\lim\limits_{k\to\infty}\|u_{n_k} - y_{n_k}\| = 0$, we obtain $\{Ay_{n_k}\}$ is bounded and

$$\lim_{k\to\infty}\|Au_{n_k} - Ay_{n_k}\| = 0.$$

Combining (3.3) and (3.4), we find that

$$\liminf_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \geq 0.$$

To prove $z \in \mathrm{VI}(C, A)$, we choose a positive decreasing sequence $\{\varepsilon_k\}$ with $\varepsilon_k \to 0$. For each k, we denote by N_k the smallest positive integer such that

$$\langle Ay_{n_j}, x - y_{n_j}\rangle + \varepsilon_k \geq 0, \forall j \geq N_k.$$

Since $\{y_{N_k}\} \subset C$, we may suppose that $Ay_{N_k} \neq 0$, for each k (otherwise, y_{N_k} is a solution). Let $d_{N_k} = \dfrac{Ay_{N_k}}{\|Ay_{N_k}\|^2}$, it follows that $\langle Ay_{N_k}, d_{N_k}\rangle = 1$. Thus

$$\langle Ay_{N_k}, x + \varepsilon_k d_{N_k} - y_{N_k}\rangle = \langle Ay_{N_k}, x - y_{N_k}\rangle + \langle Ay_{N_k}, \varepsilon_k d_{N_k}\rangle = \langle Ay_{N_k}, x - y_{N_k}\rangle + \varepsilon_k \geq 0.$$

By the pseudomonotonicity of A, we have

$$\langle A(x + \varepsilon_k d_{N_k}), x + \varepsilon_k d_{N_k} - y_{N_k}\rangle \geq 0.$$

This implies that

$$\langle Ax, x - y_{N_k}\rangle \geq \langle Ax - A(x + \varepsilon_k d_{N_k}), x + \varepsilon_k d_{N_k} - y_{N_k}\rangle - \varepsilon_k\langle Ax, d_{N_k}\rangle. \tag{3.5}$$

Next, we prove that $\lim\limits_{k\to\infty}\varepsilon_k d_{N_k} = 0$. From $u_{n_k} \rightharpoonup z$ and $\|u_{n_k} - y_{n_k}\| \to 0$, we obtain $y_{n_k} \rightharpoonup z$ as $k \to \infty$. We may suppose $Az \neq 0$ (otherwise, z is a solution). Due to (2.1), we have

$$0 < \|Az\| \leq \liminf_{k\to\infty}\|Ay_{n_k}\|.$$

From $\{y_{N_k}\} \subset \{y_{n_k}\}$ and $\varepsilon_k \to 0$ as $k \to \infty$, we obtain

$$0 \leq \limsup_{k\to\infty}\|\varepsilon_k d_{N_k}\| = \limsup_{k\to\infty}\left(\frac{\varepsilon_k}{\|Ay_{N_k}\|}\right) \leq \frac{\limsup\limits_{k\to\infty}\varepsilon_k}{\liminf\limits_{k\to\infty}\|Ay_{n_k}\|} = 0,$$

It follows that $\lim\limits_{k\to\infty}\varepsilon_k d_{N_k} = 0$. Letting $k \to \infty$, we get the right hand side of (3.5) tends to 0 because A is Lipschitz continuous, $\{y_{N_k}\}$ and $\{d_{N_k}\}$ are bounded and $\lim\limits_{k\to\infty}\varepsilon_k d_{N_k} = 0$. Thus we obtain

$$\liminf_{k\to\infty}\langle Ax, x - y_{N_k}\rangle \geq 0.$$

This implies that

$$\langle Ax, x - z\rangle = \lim_{k\to\infty}\langle Ax, x - y_{N_k}\rangle = \liminf_{k\to\infty}\langle Ax, x - y_{N_k}\rangle \geq 0, \forall x \in C,$$

This implies $z \in \mathrm{VI}(C, A)$ by Lemma 2.5. The proof is completed. □

Lemma 3.4. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Assume that (C1)-(C4) hold and the following conditions are satisfied:* $\lim\limits_{n\to\infty}\theta_n = 0$, $\sum_{n=1}^{\infty}\theta_n = \infty$, $\lim\limits_{n\to\infty}\frac{\alpha_n}{\theta_n}\|x_n - x_{n-1}\| = 0$, $\lim\limits_{n\to\infty}\frac{\beta_n}{\theta_n}\|x_n - x_{n-1}\| = 0$ *and* $\liminf\limits_{n\to\infty}\gamma_n(1 - \gamma_n) > 0$. *Then $\{x_n\}$ is bounded.*

Proof. Let $p \in \mathrm{VI}(C, A) \cap Fix(T)$. From the definition of $\{z_n\}$, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|y_n - \lambda_n(Ay_n - Au_n) - p\|^2 \\
&= \|y_n - p\|^2 + \lambda_n^2\|Ay_n - Au_n\|^2 - 2\lambda_n\langle y_n - p, Ay_n - Au_n\rangle \\
&= \|y_n - u_n\|^2 + \|u_n - p\|^2 + 2\langle y_n - u_n, u_n - p\rangle + \lambda_n^2\|Ay_n - Au_n\|^2 \\
&\quad - 2\lambda_n\langle y_n - p, Ay_n - Au_n\rangle \\
&= \|y_n - u_n\|^2 + \|u_n - p\|^2 - 2\langle y_n - u_n, y_n - u_n\rangle + 2\langle y_n - u_n, y_n - p\rangle \\
&\quad + \lambda_n^2\|Ay_n - Au_n\|^2 - 2\lambda_n\langle y_n - p, Ay_n - Au_n\rangle \\
&= \|u_n - p\|^2 - \|y_n - u_n\|^2 + 2\langle y_n - u_n + \lambda_n Au_n, y_n - p\rangle \\
&\quad + \lambda_n^2\|Ay_n - Au_n\|^2 - 2\lambda_n\langle Ay_n, y_n - p\rangle.
\end{aligned} \tag{3.6}$$

Due to $p \in \mathrm{VI}(C, A)$, we have $\langle Ap, y-p\rangle \geq 0, \forall y \in C$. By the pseudomonotonicity of A, $\langle Ay, y-p\rangle \geq 0$. Using the fact $\{y_n\} \subset C$, we get

$$\langle Ay_n, y_n - p\rangle \geq 0. \tag{3.7}$$

From Lemma 2.1 and $y_n = P_C(u_n - \lambda_n Au_n)$, we have

$$\langle y_n - u_n + \lambda_n Au_n, y_n - p\rangle \leq 0. \tag{3.8}$$

Note that $\sum_{n=0}^{\infty} \xi_n$ converges and $0 < \eta_1 < \eta_0 < \sigma < 1$. Then there exists $N_1 \in \mathbb{N}$ such that

$$\xi_n < \frac{\sigma}{\eta_0} - 1, \ \forall n \geq N_1.$$

If $\|Ay_n - Au_n\| > \frac{\eta_0}{\lambda_n}\|y_n - u_n\|$, then $\lambda_{n+1} = \eta_1 \frac{\|y_n - u_n\|}{\|Ay_n - Au_n\|}$. It follows that

$$\begin{aligned}
\lambda_n\|Ay_n - Au_n\| &= \frac{\lambda_n\lambda_{n+1}}{\lambda_{n+1}}\|Ay_n - Au_n\| = \frac{\lambda_n}{\lambda_{n+1}}\eta_1\|Ay_n - Au_n\| \\
&\leq \eta_1\|y_n - u_n\| < \sigma\|y_n - u_n\|.
\end{aligned}$$

If $\|Ay_n - Au_n\| \leq \frac{\eta_0}{\lambda_n}\|y_n - u_n\|$, then for all $n \geq N_1$, we have

$$\begin{aligned}
\lambda_n\|Ay_n - Au_n\| &= \frac{\lambda_n\lambda_{n+1}}{\lambda_{n+1}}\|Ay_n - Au_n\| = \frac{\lambda_n}{\lambda_{n+1}}(1+\xi_n)\lambda_n\|Ay_n - Au_n\| \\
&\leq \frac{\lambda_n}{\lambda_{n+1}}(1+\xi_n)\eta_0\|y_n - u_n\| \leq (1+\xi_n)\eta_0\|y_n - u_n\| < \sigma\|y_n - u_n\|.
\end{aligned}$$

Using (3.6)-(3.8), we have

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|u_n - p\|^2 - \|y_n - u_n\|^2 + \lambda_n^2\|A(y_n) - A(u_n)\|^2 \\
&\leq \|u_n - p\|^2 - \|y_n - u_n\|^2 + \sigma^2\|y_n - u_n\|^2 \\
&= \|u_n - p\|^2 - (1-\sigma^2)\|y_n - u_n\|^2,
\end{aligned} \tag{3.9}$$

which implies that $\|z_n - p\| \leq \|u_n - p\|$.

Let $t_n := \gamma_n T z_n + (1-\gamma_n) z_n$. Using Lemma 2.2, the quasi-nonexpansiveness of T and (3.9), we obtain

$$\begin{aligned}
\|t_n - p\|^2 &= \|\gamma_n T z_n + (1-\gamma_n) z_n - p\|^2 \\
&= \|\gamma_n (T z_n - p) + (1-\gamma_n)(z_n - p)\|^2 \\
&= \gamma_n \|T z_n - p\|^2 + (1-\gamma_n)\|z_n - p\|^2 - \gamma_n(1-\gamma_n)\|z_n - T z_n\|^2 \\
&\le \gamma_n \|z_n - p\|^2 + (1-\gamma_n)\|z_n - p\|^2 - \gamma_n(1-\gamma_n)\|z_n - T z_n\|^2 \\
&= \|z_n - p\|^2 - \gamma_n(1-\gamma_n)\|z_n - T z_n\|^2 \\
&\le \|u_n - p\|^2 - (1-\sigma^2)\|y_n - u_n\|^2 - \gamma_n(1-\gamma_n)\|z_n - T z_n\|^2, \qquad (3.10)
\end{aligned}$$

which implies $\|t_n - p\| \le \|u_n - p\|$.

On the other hand, from $w_n = x_n + \alpha_n(x_n - x_{n-1})$, we have

$$\begin{aligned}
\|w_n - p\| &= \|x_n - p + \alpha_n(x_n - x_{n-1})\| \\
&\le \|x_n - p\| + \alpha_n \|x_n - x_{n-1}\| \\
&= \|x_n - p\| + \theta_n \cdot \frac{\alpha_n}{\theta_n}\|x_n - x_{n-1}\|.
\end{aligned}$$

Since $\lim\limits_{n\to\infty} \frac{\alpha_n}{\theta_n}\|x_n - x_{n-1}\| = 0$, there exists $M_0 > 0$ such that $\frac{\alpha_n}{\theta_n}\|x_n - x_{n-1}\| \le M_0, \forall n \ge 1$. It follows that

$$\|w_n - p\| \le \|x_n - p\| + \theta_n M_0. \qquad (3.11)$$

Similarly, there exists $M_1 > 0$ such that $\frac{\beta_n}{\theta_n}\|x_n - x_{n-1}\| \le M_1, \forall n \ge 1$, and so

$$\|u_n - p\| \le \|x_n - p\| + \theta_n M_1. \qquad (3.12)$$

Combining (3.11) and (3.12), we see that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\theta_n f(w_n) + (1-\theta_n) t_n - p\| \\
&= \|\theta_n (f(w_n) - p) + (1-\theta_n)(t_n - p)\| \\
&\le \theta_n \|f(w_n) - f(p)\| + (1-\theta_n)\|t_n - p\| + \theta_n \|f(p) - p\| \\
&\le \theta_n \delta \|w_n - p\| + (1-\theta_n)\|t_n - p\| + \theta_n \|f(p) - p\| \\
&\le \theta_n \delta \|x_n - p\| + \theta_n^2 \delta M_0 + (1-\theta_n)\|x_n - p\| + (1-\theta_n)\theta_n M_1 + \theta_n \|f(p) - p\| \\
&\le \theta_n \delta \|x_n - p\| + \theta_n M_0 + (1-\theta_n)\|x_n - p\| + \theta_n M_1 + \theta_n \|f(p) - p\| \\
&= (1 - \theta_n(1-\delta))\|x_n - p\| + \theta_n(1-\delta) \cdot \frac{M_0 + M_1 + \|f(p) - p\|}{1-\delta} \\
&\le \max\{\|x_n - p\|, \frac{M_0 + M_1 + \|f(p) - p\|}{1-\delta}\} \\
&\le \cdots \le \max\{\|x_0 - p\|, \frac{M_0 + M_1 + \|f(p) - p\|}{1-\delta}\}.
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded, and so $\{u_n\}$, $\{w_n\}$, $\{z_n\}$, $\{f(w_n)\}$, and $\{t_n\}$ are all bounded. The proof is completed. □

Theorem 3.5. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Assume that (C1)-(C4) hold and the following conditions are satisfied: $\lim\limits_{n\to\infty} \theta_n = 0$, $\sum_{n=1}^{\infty} \theta_n = \infty$, $\lim\limits_{n\to\infty} \frac{\alpha_n}{\theta_n}\|x_n - x_{n-1}\| = 0$, $\lim\limits_{n\to\infty} \frac{\beta_n}{\theta_n}\|x_n - x_{n-1}\| = 0$ and $\liminf\limits_{n\to\infty} \gamma_n(1-\gamma_n) > 0$. Then $\{x_n\}$ converges strongly to an element $p \in \mathrm{VI}(C, A) \cap Fix(T)$, i.e., $\|x_n - p\|^2$ converges to zero, where $p = P_{\mathrm{VI}(C,A)\cap Fix(T)} f(p)$.*

Proof. The proof of the theorem is divided into the following three steps.
Step 1: From the definition of $\{x_n\}$ and Lemma 2.2, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\theta_n f(w_n)+(1-\theta_n)t_n-p\|^2\\ &= \theta_n\|(f(w_n)-p)\|^2+(1-\theta_n)\|(t_n-p)\|^2-\theta_n(1-\theta_n)\|(f(w_n)-t_n)\|^2\\ &\le \theta_n\|f(w_n)-f(p)+f(p)-p\|^2+(1-\theta_n)\|t_n-p\|^2\\ &\le \theta_n\big[\|f(w_n)-f(p)\|^2+2\langle f(p)-p,f(w_n)-p\rangle\big]+(1-\theta_n)\|t_n-p\|^2\\ &\le \theta_n\big[\delta^2\|w_n-p\|^2+2\|f(p)-p\|\|f(w_n)-p\|\big]+(1-\theta_n)\|t_n-p\|^2\\ &< \theta_n\|w_n-p\|^2+(1-\theta_n)\|t_n-p\|^2+2\theta_n\|f(p)-p\|\|f(w_n)-p\|\\ &\le \theta_n\|w_n-p\|^2+(1-\theta_n)\|t_n-p\|^2+\theta_n M_2,\end{aligned}$$

where $M_2=\sup_{n\ge1}(2\|f(p)-p\|\|f(w_n)-p\|)$. By (3.10), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 \le&\theta_n\|w_n-p\|^2+\theta_n M_2\\ &+(1-\theta_n)\big[\|u_n-p\|^2-(1-\sigma^2)\|y_n-u_n\|^2-\gamma_n(1-\gamma_n)\|z_n-Tz_n\|^2\big]\\ =&\theta_n\|w_n-p\|^2+(1-\theta_n)\|u_n-p\|^2+\theta_n M_2-(1-\theta_n)(1-\sigma^2)\|y_n-u_n\|^2\\ &-(1-\theta_n)\gamma_n(1-\gamma_n)\|z_n-Tz_n\|^2.\end{aligned}$$

Using (3.11), we get

$$\begin{aligned}\|w_n-p\|^2 &\le (\|x_n-p\|+\theta_n M_0)^2\\ &= \|x_n-p\|^2+\theta_n(2M_0\|x_n-p\|+\theta_n M_0^2)\\ &\le \|x_n-p\|^2+\theta_n M_3,\end{aligned}$$

where $M_3=\sup_{n\ge1}(2M_0\|x_n-p\|+M_0^2)$. Similarly, from (3.12), we obtain

$$\begin{aligned}\|u_n-p\|^2 &\le (\|x_n-p\|+\theta_n M_1)^2\\ &= \|x_n-p\|^2+\theta_n(2M_1\|x_n-p\|+\theta_n M_1^2)\\ &\le \|x_n-p\|^2+\theta_n M_4,\end{aligned}$$

where $M_4=\sup_{n\ge1}(2M_1\|x_n-p\|+M_1^2)$. It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \theta_n\big[\|x_n-p\|^2+\theta_n M_3\big]+(1-\theta_n)\big[\|x_n-p\|^2+\theta_n M_4\big]\\ &\quad+\theta_n M_2-(1-\theta_n)(1-\sigma^2)\|y_n-u_n\|^2-(1-\theta_n)\gamma_n(1-\gamma_n)\|z_n-Tz_n\|^2\\ &\le \|x_n-p\|^2-(1-\theta_n)(1-\sigma^2)\|y_n-u_n\|^2-(1-\theta_n)\gamma_n(1-\gamma_n)\|z_n-Tz_n\|^2+\theta_n M_5,\end{aligned}$$

where $M_5=M_2+M_3+M_4$. Therefore,

$$\begin{aligned}&(1-\theta_n)(1-\sigma^2)\|y_n-u_n\|^2+(1-\theta_n)\gamma_n(1-\gamma_n)\|z_n-Tz_n\|^2\\ &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\theta_n M_5.\end{aligned}\tag{3.13}$$

Step 2: From $w_n=x_n+\alpha_n(x_n-x_{n-1})$, we have

$$\begin{aligned}\|w_n-p\|^2 &= \|x_n+\alpha_n(x_n-x_{n-1})-p\|^2\\ &\le \|x_n-p\|^2+2\alpha_n\langle x_n-x_{n-1},w_n-p\rangle\\ &\le \|x_n-p\|^2+2\alpha_n\|x_n-x_{n-1}\|\|w_n-p\|\\ &\le \|x_n-p\|^2+\alpha_n\|x_n-x_{n-1}\|M_6,\end{aligned}$$

where $M_6 = \sup_{n\geq 1}(2\|w_n - p\|)$. By $\|t_n - p\| \leq \|u_n - p\|$, we have

$$\begin{aligned}
\|t_n - p\|^2 &\leq \|u_n - p\|^2 \\
&= \|x_n + \beta_n(x_n - x_{n-1}) - p\|^2 \\
&\leq \|x_n - p\|^2 + 2\beta_n\langle x_n - x_{n-1}, u_n - p\rangle \\
&\leq \|x_n - p\|^2 + 2\beta_n\|x_n - x_{n-1}\|\|u_n - p\| \\
&\leq \|x_n - p\|^2 + \beta_n\|x_n - x_{n-1}\|M_7,
\end{aligned}$$

where $M_7 = \sup_{n\geq 1}(2\|u_n - p\|)$. Therefore

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\theta_n f(w_n) + (1-\theta_n)t_n - p\|^2 \\
&= \|\theta_n(f(w_n) - f(p)) + (1-\theta_n)(t_n - p) + \theta_n(f(p) - p)\|^2 \\
&\leq \|\theta_n(f(w_n) - f(p)) + (1-\theta_n)(t_n - p)\|^2 + 2\theta_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\leq \theta_n\delta^2\|w_n - p\|^2 + (1-\theta_n)\|t_n - p\|^2 + 2\theta_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\leq \theta_n\delta(\|x_n - p\|^2 + \alpha_n\|x_n - x_{n-1}\|M_6) + (1-\theta_n)(\|x_n - p\|^2 + \beta_n\|x_n - x_{n-1}\|M_7) \\
&\quad + 2\theta_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\leq \theta_n\delta\|x_n - p\|^2 + \alpha_n\|x_n - x_{n-1}\|M_6 + (1-\theta_n)\|x_n - p\|^2 + \beta_n\|x_n - x_{n-1}\|M_7 \\
&\quad + 2\theta_n\langle f(p) - p, x_{n+1} - p\rangle \\
&= (1 - \theta_n(1-\delta))\|x_n - p\|^2 + \theta_n(1-\delta)\big[\frac{M_6\alpha_n + M_7\beta_n}{\theta_n(1-\delta)}\|x_n - x_{n-1}\| \\
&\quad + \frac{2}{1-\delta}\langle f(p) - p, x_{n+1} - p\rangle\big].
\end{aligned} \tag{3.14}$$

Step 3: We discuss the following two cases.

Case 1. If there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2$, then $\lim\limits_{n\to\infty}\|x_n - p\|^2$ exists. By the definitions of $\{\alpha_n\}$, $\{\beta_n\}$ and (3.13),

$$\lim_{n\to\infty}\|u_n - y_n\| = 0, \tag{3.15}$$

$$\lim_{n\to\infty}\|z_n - Tz_n\| = 0,$$

$$\lim_{n\to\infty}\|x_n - u_n\| = \lim_{n\to\infty}\beta_n\|x_n - x_{n-1}\| = \lim_{n\to\infty}\theta_n \cdot \frac{\beta_n}{\theta_n}\|x_n - x_{n-1}\| = 0. \tag{3.16}$$

Due to the prove of Lemma 3.4, we have that $\|y_n - z_n\| = \lambda_n\|Ay_n - Au_n\| < \sigma\|y_n - u_n\|$. From (3.15), we obtain

$$\lim_{n\to\infty}\|y_n - z_n\| = 0.$$

It follows that

$$\lim_{n\to\infty}\|z_n - u_n\| = \lim_{n\to\infty}\|z_n - y_n + y_n - u_n\| \leq \lim_{n\to\infty}\|z_n - y_n\| + \lim_{n\to\infty}\|y_n - u_n\| = 0.$$

This together with (3.16) yields

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

Note that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \|\theta_n f(w_n) - \theta_n t_n\| + \|\gamma_n Tz_n + (1-\gamma_n)z_n - x_n\| \\
&= \theta_n\|f(w_n) - t_n\| + \|\gamma_n(Tz_n - z_n) + (z_n - x_n)\| \\
&\leq \theta_n(\|f(w_n) - p\| + \|t_n - p\|) + \gamma_n\|Tz_n - z_n\| + \|z_n - x_n\|.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $x_{n_k}\rightharpoonup z\in H$, and thus

$$\limsup_{n\to\infty}\langle f(p)-p,x_n-p\rangle=\lim_{k\to\infty}\langle f(p)-p,x_{n_k}-p\rangle=\langle f(p)-p,z-p\rangle.$$

From $x_{n_k}\rightharpoonup z$, (3.12), (3.13) and Lemma 3.2, we obtain $z\in \mathrm{VI}(C,A)$. Since $\lim\limits_{n\to\infty}\|z_n-x_n\|=0$, it follows that $z_{n_k}\rightharpoonup z$. Moreover, from $\lim\limits_{n\to\infty}\|z_n-Tz_n\|=0$ and the demiclosedness of $I-T$, we have $z\in F(T)$, and so $z\in \mathrm{VI}(C,A)\cap F(T)$. According to Lemma 2.1 and $p=P_{\mathrm{VI}(C,A)\cap Fix(T)}f(p)$, we derive

$$\limsup_{n\to\infty}\langle f(p)-p,x_n-p\rangle=\langle f(p)-p,z-p\rangle\le 0,$$

Therefore

$$\limsup_{n\to\infty}\langle f(p)-p,x_{n+1}-p\rangle\le\limsup_{n\to\infty}\langle f(p)-p,x_{n+1}-x_n\rangle+\limsup_{n\to\infty}\langle f(p)-p,x_n-p\rangle\le 0.$$

By (3.14) and Lemma 2.7, $\lim\limits_{n\to\infty}\|x_n-p\|^2=0$, i.e. $x_n\to p$ as $n\to\infty$.

Case 2. If there exists a subsequence $\{\|x_{n_j}-p\|^2\}$ of $\{\|x_n-p\|^2\}$ such that $\|x_{n_j}-p\|^2\le\|x_{n_j+1}-p\|^2,\forall j\in\mathbb{N}$, then by Lemma 2.8, there exists a nondecreasing sequence of integers $\{m_k\}$ such that $\lim\limits_{k\to\infty}m_k=\infty$ and for all $k\in\mathbb{N}$, we have

$$\|x_{m_k}-p\|^2\le\|x_{m_k+1}-p\|^2,\ \|x_k-p\|^2\le\|x_{m_k+1}-p\|^2.$$

Using (3.13), we obtain

$$\begin{aligned}&(1-\theta_{m_k})(1-\sigma^2)\|y_{m_k}-u_{m_k}\|^2+\gamma_{m_k}(1-\gamma_{m_k})(1-\theta_{m_k})\|z_{m_k}-Tz_{m_k}\|^2\\&\le\|x_{m_k}-p\|^2-\|x_{m_k+1}-p\|^2+\theta_{m_k}M_5\le\theta_{m_k}M_5,\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\|y_{m_k}-u_{m_k}\|=0,\ \lim_{k\to\infty}\|z_{m_k}-Tz_{m_k}\|=0.$$

Following the similar arguments as in Case 1, we have

$$\lim_{k\to\infty}\|x_{m_k}-z_{m_k}\|=0,\ \lim_{k\to\infty}\|x_{m_k+1}-x_{m_k}\|=0,$$

$$\limsup_{k\to\infty}\langle f(p)-p,x_{m_k+1}-p\rangle\le 0.$$

From (3.14), we have

$$\begin{aligned}&\|x_{m_k+1}-p\|^2\\&\le(1-\theta_{m_k}(1-\delta))\|x_{m_k}-p\|^2\\&\quad+\theta_{m_k}(1-\delta)\big[\frac{M_6\alpha_{m_k}+M_7\beta_{m_k}}{\theta_{m_k}(1-\delta)}\|x_{m_k}-x_{m_k-1}\|+\frac{2}{1-\delta}\langle f(p)-p,x_{m_k+1}-p\rangle\big]\\&\le(1-\theta_{m_k}(1-\delta))\|x_{m_k+1}-p\|^2\\&\quad+\theta_{m_k}(1-\delta)\big[(\frac{M_6}{1-\delta}\cdot\frac{\alpha_{m_k}}{\theta_{m_k}}+\frac{M_7}{1-\delta}\cdot\frac{\beta_{m_k}}{\theta_{m_k}})\|x_{m_k}-x_{m_k-1}\|+\frac{2}{1-\delta}\langle f(p)-p,x_{m_k+1}-p\rangle\big],\end{aligned}$$

It follows that

$$\|x_{m_k+1}-p\|^2\le(\frac{M_6}{1-\delta}\cdot\frac{\alpha_{m_k}}{\theta_{m_k}}+\frac{M_7}{1-\delta}\cdot\frac{\beta_{m_k}}{\theta_{m_k}})\|x_{m_k}-x_{m_k-1}\|+\frac{2}{1-\delta}\langle f(p)-p,x_{m_k+1}-p\rangle),$$

Since $\|x_k - p\|^2 \leq \|x_{m_k+1} - p\|^2$, one has

$$\limsup_{k\to\infty} \|x_k - p\|^2 \leq 0.$$

That is $x_n \to p$. The proof is completed. □

4. Numerical Experiments

In this section, two numerical examples are given to illustrate the performances of Algorithm 1 (shortly, Alg 1) and compare it with Algorithm 1 in [14] (shortly, SCM), Algorithm 3.1 in [36] (shortly, DIMTEM) and Algorithm 3 in [19] (shortly, DISEM). All the codes are written in Matlab R2020b and performed on PC Desktop Intel Core(TM) i5-1135G7 @ 2.40GHz RAM 8.00 GB. “Iter” represents the number of iterations and “Time” represents the iteration time, the unit is a second.

In numerical testing, we employ the sequence $E_n = \|x_n - x^*\|$ to quantify the error at the n-th iteration, where x_n is the current iterate and x^* is a solution of the problem. The iteration is terminated when the error satisfies the stopping rule $E_n < err$ for all tested algorithms. We also take $f(x) = 0.5x$. For our algorithm, the parameters α_n and β_n can be chosen as follows:

$$\alpha_n = \begin{cases} \min\left\{\frac{\delta_n}{\|x_n - x_{n-1}\|}, \alpha\right\}, & \text{if } x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise}, \end{cases}$$

$$\beta_n = \begin{cases} \min\left\{\frac{\delta_n}{\|x_n - x_{n-1}\|}, \beta\right\}, & \text{if } x_n \neq x_{n-1}, \\ \beta, & \text{otherwise}, \end{cases}$$

where α and β are constants such that $0 < \alpha < 1$, $0 < \beta < 1$ and $\{\delta_n\}$ is a positive sequence such that $\lim\limits_{n\to\infty} \frac{\delta_n}{\theta_n} = 0$.

Example 4.1. Consider the linear mapping $A : R^m \to R^m$ such that

$$A(x) = Mx + q,$$

where $q \in R^m$ and $M = N^T N + S + D$, N is an $m \times m$ matrix, S is an $m \times m$ skew-symmetric matrix and D is an diagonal matrix that diagonal entries are nonnegative (so M is positive semi-definite). The feasible set C is defined as $C := \{x \in R^m : Bx \leq b\}$, where B is a $k \times m$ random matrix and b is a random vector with nonnegative. It is easy to see that A is monotone and L-Lipschitz continuous with Lipschitz constant $L = \|M\|$. Let $T(x) = 0.5x$ and $q = 0$. We also observe $x^* = (0, \cdots, 0)^T$ is the unique solution of problem (1.2) with respect to A, C and T.

The parameters are selected as follows:

Alg 1: $\alpha = 0.1$, $\beta = 0.4$, $\lambda_1 = 0.1$, $\eta_1 = 0.4$, $\eta_0 = 0.5$, $\theta_n = \frac{1}{n+1}$, $\delta_n = \frac{1}{(n+1)^2}$, $\gamma_n = \frac{n}{2n+1}$ and $\xi_n = \frac{1}{(n+1)^3}$.

SCM: $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{24(n+1)}$, $\lambda_1 = 0.1$, $\mu = 0.4$, $\gamma_n = \frac{n}{2n+1}$ and $\rho_n = \frac{1}{(n+1)^3}$.

DIMTEM: $\alpha = 0.02$, $\beta_n = \frac{n}{24(n+1)}$, $\lambda_1 = 0.1$, $\mu = 0.4$, $\gamma_n = \frac{n}{3.1(n+1)}$ and $\rho_n = \frac{1}{(n+1)^3}$.

DISEM: $\alpha = 0.1$, $\beta = 0.4$, $\tau = 1.4$, $\lambda_1 = 0.1$, $\mu = 0.4$, $\gamma_n = 0.25$ and $\rho_n = \frac{1}{(n+1)^3}$.

Let $k = 10$, $x_0 = x_1 = (1, \ldots, 1)^T$, and N, S and D be randomly generated. The stopping criterion is $\|x_n - x^*\| < err = 10^{-10}$. The numerical results are described in Table 1, Table 2 and Figure 1.

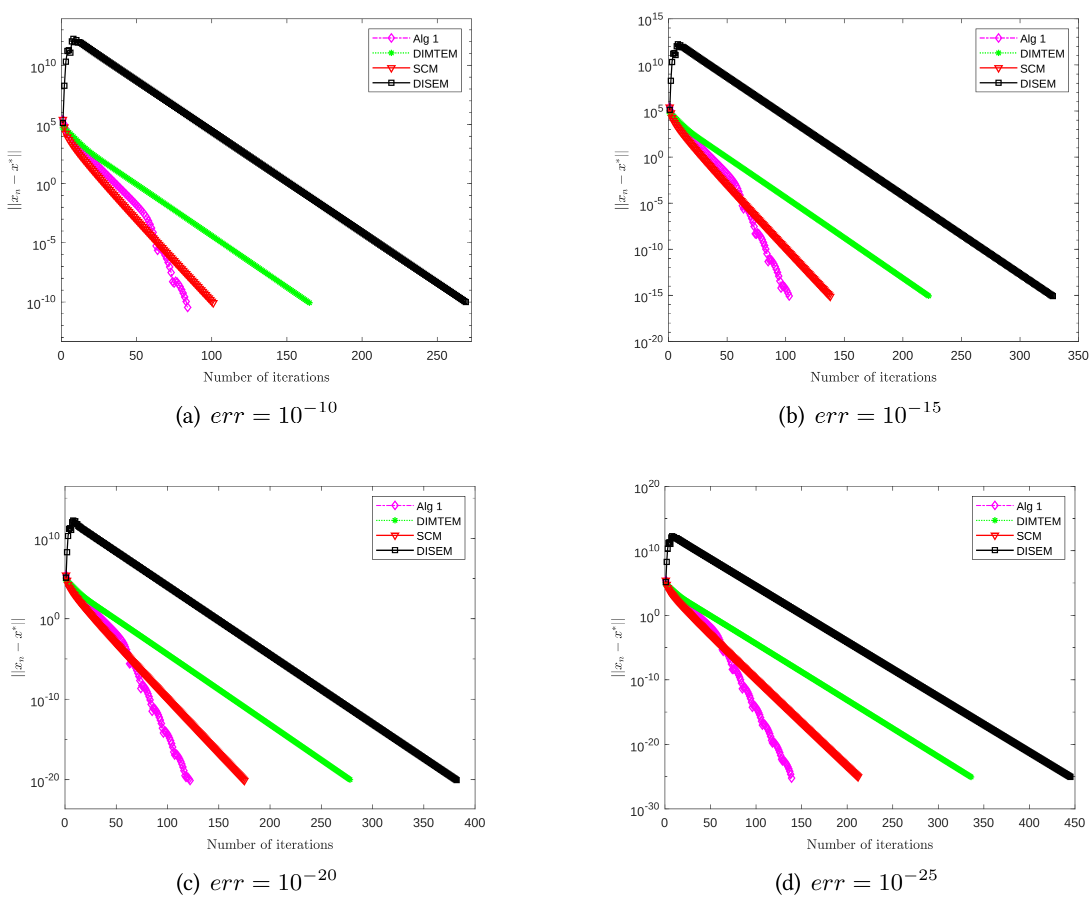
From Table 1, Table 2 and Figure 1, it can be observed that under different m and err, Algorithm 1 outperforms the algorithms SCM, DIMTEM and DISEM in terms of running time and number of iterations.

TABLE 1. Numerical results for Example 1 with different m and $err = 10^{-10}$

	m=20		m=50		m=100		m=150	
	Iter	Time	Iter	Time	Iter	Time	Iter	Time
Alg 1	54	0.1831	71	0.6430	83	0.8075	91	1.1634
SCM	75	0.3357	90	0.9189	101	1.0186	107	1.2873
DIMTEM	142	0.6714	166	1.9556	183	3.7758	174	4.4704
DISEM	181	1.2827	308	2.6140	327	3.3547	294	8.0670

TABLE 2. Numerical results for Example 1 with different err and $m = 100$

	$err = 10^{-10}$		$err = 10^{-15}$		$err = 10^{-20}$		$err = 10^{-25}$	
	Iter	Time	Iter	Time	Iter	Time	Iter	Time
Alg 1	84	0.6280	103	0.7296	123	0.7846	139	0.9708
SCM	101	1.0734	138	1.4604	175	1.8776	212	2.2703
DIMTEM	183	2.9826	247	4.6638	311	4.3232	375	4.7183
DISEM	327	3.2647	400	4.5301	471	5.5882	544	6.1253



(a) $err = 10^{-10}$
(b) $err = 10^{-15}$
(c) $err = 10^{-20}$
(d) $err = 10^{-25}$

FIGURE 1. Comparison of algorithms for Example 1 with different err and $m = 100$

Example 4.2. Let $A : R^3 \to R^3$ be defined by

$$A(x) = (e^{-\|x\|^2} + \beta)Mx,$$

where $\beta = 0.2$ and

$$M = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1.5 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

The feasible set C is defined as $C := \{x = (x^1, x^2, x^3) \in [-5, 5]^3 : x^1+x^2+x^3 = 0\}$. Let $T(x) = 0.4x$. Observe that $S = \{(0, 0, 0)\}$ with respect to A, C and T. It can be seen that A are Lipschitz continuous and pseudo-monotone but not monotone (Bot et al. 2020).

The parameters are selected as follows:

Alg 1: $\alpha = 0.1$, $\beta = 0.3$, $\lambda_1 = 0.5$, $\eta_1 = 0.4$, $\eta_0 = 0.5$, $\theta_n = \frac{1}{n+1}$, $\delta_n = \frac{1}{(n+1)^2}$, $\gamma_n = \frac{n}{2n+1}$ and $\xi_n = \frac{1}{(n+1)^3}$.

SCM: $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{n}{24(n+2)}$, $\lambda_1 = 0.5$, $\mu = 0.4$, $\gamma_n = \frac{n}{2n+1}$ and $\rho_n = \frac{1}{(n+1)^3}$.

DIMTEM: $\alpha = 0.02$, $\beta_n = \frac{n}{24(n+2)}$, $\lambda_1 = 0.5$, $\mu = 0.4$, $\gamma_n = \frac{n}{3.1(n+1)}$ and $\rho_n = \frac{1}{(n+1)^3}$.

DISEM: $\alpha = 0.1$, $\beta = 0.3$, $\tau = 1.4$, $\lambda_1 = 0.5$, $\mu = 0.4$, $\gamma_n = 0.25$ and $\rho_n = \frac{1}{(n+1)^3}$.

We adopt the initial points $x_0 = (1, 1, 1)$ and $x_1 = (2, 2, 2)$. In this example, we compare the performances of four algorithms with different err. The numerical results are presented in Table 3 and Figure 2.

TABLE 3. Numerical results for Example 2 with different err

	$err = 10^{-10}$		$err = 10^{-15}$		$err = 10^{-20}$		$err = 10^{-25}$	
	Iter	Time	Iter	Time	Iter	Time	Iter	Time
Alg 1	26	0.0541	40	0.0701	49	0.0880	57	0.1003
SCM	36	0.0642	58	0.1081	65	0.1232	73	0.1705
DIMTEM	94	0.1386	141	0.2102	175	0.2646	210	0.3861
DISEM	101	0.1797	152	0.2967	189	0.4436	225	0.6358

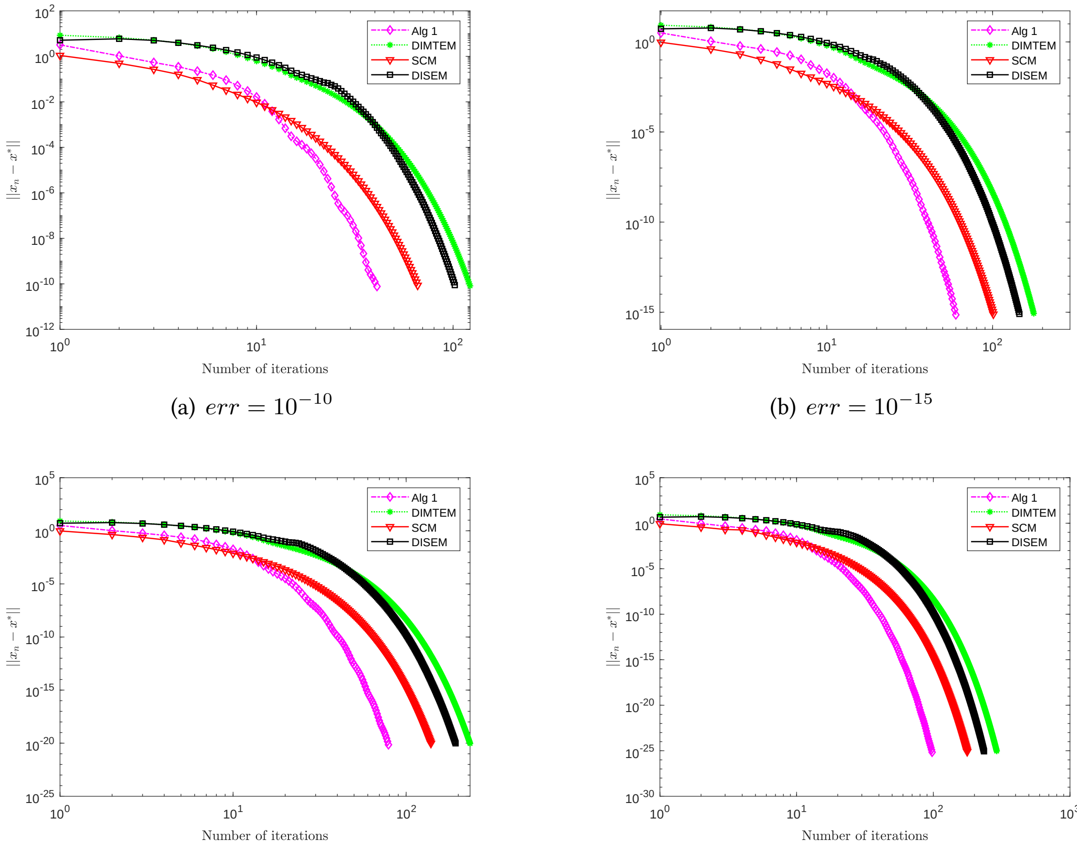
As shown in Table 3 and Figure 2, Algorithm 1 outperforms the algorithms SCM, DISEM and DIMTEM on different err. This result affirms that our algorithm has cheaper computational loads than the algorithms SCM, DISEM and DIMTEM.

5. Concluding Remarks

In this paper, a double inertial Tseng extragradient algorithm with self-adaptive stepsize is proposed to solve a pseudomonotone variational inequality and a fixed point problem with a quasi-nonexpansive mapping in Hilbert spaces. Under some standard conditions, a strong convergence theorem is obtained. Numerical experiments illustrate the performances and advantages of our algorithm. In future works, we will relax the psedudomonotonicity of A to quasi-monotonicity, and drop the assumptions that $\lim\limits_{n\to\infty} \frac{\alpha_n}{\theta_n}\|x_n - x_{n-1}\| = 0$ and $\lim\limits_{n\to\infty} \frac{\beta_n}{\theta_n}\|x_n - x_{n-1}\| = 0$.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.



(a) $err = 10^{-10}$ (b) $err = 10^{-15}$ (c) $err = 10^{-20}$ (d) $err = 10^{-25}$

FIGURE 2. Comparison of algorithms for Example 2 with different err and $m = 100$

Acknowledgments

This work was supported by the NSF of Chongqing (CSTB2024NSCQ-MSX1282), the Education Committee Project Research Foundation of Chongqing (KJZD-K202500805), Chongqing Social Science Planning Project (2024NDYB100) and the Team Building Project for Graduate Tutors in Chongqing (yds223010).

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