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VISCOSITY-INERTIAL APPROXIMATION METHOD FOR ATTRACTIVE POINTS OF WIDELY MORE GENERALIZED HYBRID MAPPINGS

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ABSTRACT. This research presents a viscosity-inertial iterative scheme for approximating attractive point of finite family widely more generalized hybrid mapping in Hilbert space. The proposed method integrates terms into the viscosity framework to accelerate convergence while maintaining strong convergence guarantee. The scheme is established under less restrictive assumptions. A strong convergence theorem is proved under appropriate control conditions on the parameter sequences.

Keywords. Strong convergence, Widely more generalized hybrid mapping, Attractive point, Viscosity-inertial approximation method.

1. INTRODUCTION

Let H be a real Hilbert space and C a nonempty closed and convex subset of H. Also T be a mapping from C into H, the set of fixed points of T is denoted by $F(T)$= $\{x \in C : Tx = x\}$.
A mapping $T : C \to H$ is called

(1) Nonexpansive mapping if $K = 1$ that is $\|Tx - Ty\| \leq \|x - y\|$.
(2) Widely more generalized hybrid mapping if there exists $\alpha, \beta, \gamma, \sigma, \epsilon, \zeta$ and ω and such that
$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \sigma\|x - y\|^2 + \epsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2$
$+ \omega\| (x - Tx) - (y - Ty) \|^2 \leq 0 \, \forall \, x, y \in C.$

A point $u \in X$ is called an attractive point if it satisfies the following condition
$\|Tx - u\| \leq \|x - u\|$ for every $x \in C$. Let $T_1, T_2 : C \to H$, where C is a nonempty subset of H, then the set of all common attractive points for T_1 and T_2 is denoted by $A\,(T_1, T_2)$ and defined as
$A\,(T_1, T_2) = \{u \in H : \max\left(\|T_1x - u\|, \|T_2x - u\|\right) \leq \|x - u\|, \forall x \in C\}$. Moreover,
$A\,(T_1, T_2) = A\,(T_1) \cap A\,(T_2)$. For a finite family $T_1, T_2, \ldots, T_n$ of nonlinear mappings, the set of common attractive points is denoted as $A\,(T_i)$=$\left\{u \in H : \max\limits_{1 \leq i \leq n}\left(\|T_ix - u\|\right) \leq \|x - u\|, \forall x \in C\right\}$.
Takahashi and Takeuchi were the first to introduce the perception of attractive points in Hilbert spaces [9]. The main goal of the introduction was to remove the convexity and closedness assumptions commonly imposed on a nonempty subset in the well-known nonlinear ergodic theorem of Baillon [2]. They also proved an existence theorem for attractive points without the need for convexity. It is evident from the definition that, generally speaking, a fixed point need not be an attractive point and an attractive point need not be a fixed point. F(S) = $\{v \in C : Sv = v\}$ and
$A\,(S)$ = $\{v \in H : \|Sx - v\| \leq \|x - v\| x \in C\}$, where C ⊂ H and T : C → H is an operator that is nonlinear. The author [5] utilized the iterative process introduced by [11] to establish both weak and

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strong convergence results for the sequence x_n defined by

$$\begin{cases} x_1 \in C, \\ x_{k+1} = \alpha_n x_k + \beta_k T_1 x_k + \gamma_k T_2 x_k \quad \forall\, k \in N. \end{cases} \tag{1.1}$$

The authors of [4] presented an accelerated iterative algorithm for computing a common fixed point of an infinite family of nonexpansive mappings in Hilbert space with a structure similar to Nesterov's [7] acceleration principal approach which laid the foundation for modern inertial method in optimization and fixed points. More recently [8] established weak and strong convergence theorems for common attractive points of two generalized hybrid mappings, without requiring the domain to be closed. Their analysis, conducted in Hilbert space using the iterative process (1.1), achieved strong convergence results by assuming compactness of the mappings using so-called condition A, which postulates the existence of a nondecreasing function S :$[0, \infty) \to [0, \infty)$ satisfying $S(0) = 0$ and $S(x) > 0$, for any $x > 0$ ensuring that $S(x_n, d(A(T_1, T_2))) \leq \|x_n - T_1 x_n\|$ or $S(x_n, d(A(T_1, T_2))) \leq \|x_n - T_2 x_n\|$. In a related development [3] proposed a viscosity approximation method and proved a strong convergence theorem for the attractive point of a finite family of widely more generalized hybrid mappings defined by

$$\begin{cases} x_1 \in C, \\ z_n = d_{n,0} x_n \sum_{i=1}^{N} d_{n,i} T_i x_n \quad \forall\, i = 1, 2, 3, \ldots, N, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \Omega_n z_n, \quad \forall\, n \geq 1. \end{cases} \tag{1.2}$$

Motivated by the work of [3] and the studies mentioned, the present study aims to extend this result by incorporating an inertial term into the viscosity approximation in order to accelerate the convergence rate, relaxing some of the restrictive assumptions(iii) imposed in earlier study.

2. Preliminaries

In this section we give some useful definitions and lemmas that are going to be used in our work.

Lemma 2.1. [6] *Let C be a nonempty subset of a Hilbert space H. Let $T : C \to H$ be an $(\alpha, \beta, \gamma, \sigma, \epsilon, \zeta, \omega)$ -widely more generalized hybrid mapping that satisfies either of the following conditions:*

(1) $\alpha + \beta + \gamma + \sigma \geq 0$ *and* $\alpha + \gamma > 0$ *and* $\epsilon + \omega \geq 0$
(2) $\alpha + \beta + \gamma + \sigma \geq 0$ *and* $\alpha + \beta > 0$ *and* $\zeta + \omega \geq 0$
if $x_n \rightharpoonup u$ *and* $\|x_n - Tx_n\|$ *= 0 as n* $\to \infty$ *then u* $\in A(T)$.

Lemma 2.2. [10] *Let* $\{S_n\}$, $\{b_n\}$ *be positive real numbers,* $\{\sigma_n\}$ *be contained in* $[0, 1]$ *and* $\{t_n\}$ *be a sequence of real numbers such that*
$S_{n+1} \leq (1 - \sigma_n) S_n + \sigma_n t_n + b_n$,
then the following conditions hold:

i. $\sum_{i=1}^{\infty} \sigma_n = \infty$,
ii. $\sum_{i=1}^{\infty} b_n < \infty$,
iii. $\lim_{n\to\infty} \sup t_n \leq$ *0 implies* $\lim_{n\to\infty} S_n = 0, \forall$ *n* $\in$ *N.*

3. Main Results

We introduce a viscosity-inertial approximation method to prove a strong convergence theorem of widely more generalized hybrid mappings in Hilbert space.

Theorem 3.1. *Let H be a Hilbert space and C be a nonempty convex and bounded subset of H. Suppose $T_i : C \to H$ for any $i \in \{1, 2, 3, ..., N\}$ be a finite family of $(\alpha, \beta, \gamma, \sigma, \epsilon, \zeta, \omega)$-widely more generalized hybrid mappings with $\bigcap_{i=1}^{N} A(T_i) \neq \emptyset$. Let $\{y_n\}$ be a sequence defined by*

$$\begin{cases} y_0, y_1 \in C, \\ x_n = y_n + \beta_n (y_n - y_{n-1}), \quad n \geq 1, \\ z_n = d_{n,0} x_n + \sum_{i=1}^{N} d_{n,i} T_i x_n \quad \forall i = 1, 2, 3, \dots, N, \\ y_{n+1} = a_n f(y_n) + b_n x_n + c_n z_n \quad \forall\, n \geq 1, \end{cases} \tag{3.1}$$

where $f : C \to C$ is a contraction mapping with $K \in [0, 1)$, a_n, b_n, c_n, $\{d_{n,0}\}$ and $\{d_{n,i}\}$ are sequences in $(0, 1)$ with $d_{n,0} + \sum_{i=1}^{N} d_{n,i}$ = 1 and $a_n + b_n + c_n$ =1 $\forall$ $n \geq$ 1 and the following conditions hold:

(1) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$,
(2) $\lim_{n\to\infty} \beta_n \|y_n - y_{n-1}\| = 0$ *and* $\lim_{n\to\infty} \frac{\beta_n}{a_n} \|y_n - y_{n-1}\| = 0$,
(3) $0 < a \leq b_n \leq b < 1$,
(4) $0 < c \leq c_n \leq d < 1$.

Lemma 3.2. *Let H be a Hilbert space and C be a nonempty convex and bounded subset of H. Suppose $T_i i = 1^N$ is a finite family of $(\alpha, \beta, \gamma, \sigma, \epsilon, \omega, \zeta)$-widely more generalized hybrid mappings with $\bigcap_{i=1}^{N} A(T_i)$ $\neq \emptyset$. Then $\{y_n\}$ defined by (3.1) is bounded.*

Proof. Suppose $\bigcap_{i=1}^{N} A(T_i) \neq \emptyset$. Let $r \in \bigcap_{i=1}^{N} A(T_i)$ then we show that (y_n) is bounded. As defined by x_n and z_n in (3.1), we have

$$\|x_n - r\| = \|y_n + \beta_n (y_n - y_{n-1}) - r\|.$$

Thus

$$\|x_n - r\| \leq \|y_n - r\| + \beta_n \|y_n - y_{n-1}\|. \tag{3.2}$$

$$\begin{aligned} \|z_n - r\| &= \|d_{n,0} x_n + \sum_{i=1}^{N} d_{n,i} T_i x_n - r\| \\ &\leq d_{n,0} \|x_n - r\| + \sum_{i=1}^{N} d_{n,i} \|T_i x_n - r\| \\ &\leq d_{n,0} \|x_n - r\| + \sum_{i=1}^{N} d_{n,i} \|x_n - r\| \\ &= \left(d_{n,0} + \sum_{i=1}^{N} d_{n,i} \right) \|x_n - r\|. \end{aligned}$$

Therefore,

$$\|z_n - r\| \leq \|x_n - r\|. \tag{3.3}$$

From equations (3.2) and (3.3), we have

$$
\begin{aligned}
\|y_{n+1}-r\| =&\|a_n f(y_n)+b_n x_n+c_n z_n-r\|\\
=&\|a_n(f(y_n)-r)+b_n(x_n-r)+c_n(z_n-r)\|\\
\le& a_n\|f(y_n)-r\|+b_n\|x_n-r\|+c_n\|z_n-r\|\\
=&a_n\|f(y_n)-f(r)+f(r)-r\|+b_n\|x_n-r\|+c_n\|z_n-r\|\\
\le& a_n\|f(y_n)-f(r)\|+a_n\|f(r)-r\|+b_n\|x_n-r\|+c_n\|x_n-r\|\\
=&a_n\|f(y_n)-f(r)\|+(b_n+c_n)\|x_n-r\|+a_n\|f(r)-r\|\\
\le& a_n L\|y_n-r\|+(b_n+c_n)\left[\|y_n-r\|+\beta_n\|y_n-y_{n-1}\|\right]+a_n\|f(r)-r\|\\
=&a_n L\|y_n-r\|+(b_n+c_n)\|y_n-r\|+(b_n+c_n)\beta_n\|y_n-y_{n-1}\|+a_n\|f(r)-r\|\\
=&a_n L\|y_n-r\|+(1-a_n)\|y_n-r\|+(b_n+c_n)\beta_n\|y_n-y_{n-1}\|+a_n\|f(r)-r\|\\
=&\big(a_n L+(1-a_n)\big)\|y_n-r\|+(b_n+c_n)\beta_n\|y_n-y_{n-1}\|+a_n\|f(r)-r\|\\
=&\big(1-a_n(1-L)\big)\|y_n-r\|+(b_n+c_n)\beta_n\|y_n-y_{n-1}\|+a_n\|f(r)-r\|\\
=&\big(1-a_n(1-L)\big)\|y_n-r\|+\frac{a_n(1-L)}{a_n(1-L)}\Big[(b_n+c_n)\beta_n\|y_n-y_{n-1}\|+a_n\|f(r)-r\|\Big]\\
=&\big(1-a_n(1-L)\big)\|y_n-r\|\\
&+a_n(1-L)\left[\frac{b_n+c_n}{1-L}\times\frac{\beta_n}{a_n}\|y_n-y_{n-1}\|+\frac{a_n}{a_n(1-L)}\|f(r)-r\|\right].
\end{aligned}
$$

Since $\lim_{n\to\infty}\frac{\beta_n}{a_n}\|y_n-y_{n-1}\|=0$, we can find $T_1>0$ such that $\frac{\beta_n}{a_n}\|y_n-y_{n-1}\|\le T_1\ \forall n\ge 1$.

$$
\begin{aligned}
\|y_{n+1}-r\| \le&\big(1-a_n(1-L)\big)\|y_n-r\|+a_n(1-L)\left[\frac{(b+d)T_1}{1-L}+\frac{\|f(r)-r\|}{1-L}\right]\\
\le&\big(1-a_n(1-L)\big)\max\left\{\|y_n-r\|,\left[\frac{(b+d)T_1}{1-L}+\frac{\|f(r)-r\|}{1-L}\right]\right\}\\
&+a_n(1-L)\max\left\{\|y_n-r\|,\left[\frac{(b+d)T_1}{1-L}+\frac{\|f(r)-r\|}{1-L}\right]\right\}\\
\|y_{n+1}-r\| \le&\max\left\{\|y_n-r\|,\left[\frac{(b+d)T_1}{1-L}+\frac{\|f(r)-r\|}{1-L}\right]\right\}.
\end{aligned}
$$

By induction we have

$$
\|y_n-r\|\le\max\left\{\|y_0-r\|,\left[\frac{(b+d)T_1}{1-L}+\frac{\|f(r)-r\|}{1-L}\right]\right\}\quad\forall n\ge 1. \tag{3.4}
$$

By (3.4) this shows that the sequence $\{\|y_n-r\|\}$ is bounded $\forall\, r\in A(T_i)$. Therefore $\{y_n\}$ is bounded. It follows that $\{x_n\}$, $\{z_n\}$, $\{f(y_n)\}$, $\{T_i x_n\}$ are all bounded.

□

Lemma 3.3. *Assume $\{y_n\}$ is a sequence in Lemma 3.2 and let $a_n, b_n, c_n, \{d_{n,0}\}$ and $\{d_{n,i}\}$ are sequences in $(0,1)$ satisfying the assumptions outlined above, then $\lim_{n\to\infty}\|y_n-T_i y_n\|$ = 0.*

Proof. We assert the following: we pick $r\in\bigcap_{i=1}^{N}A(T_i)$,

$$
\lim_{n\to\infty}\|x_n-T_i x_n\|=0,\qquad \lim_{n\to\infty}\|y_n-z_n\|=0,\qquad \lim_{n\to\infty}\|y_n-x_n\|=0,
$$

$$\lim_{n\to\infty}\|z_n - x_n\| = 0, \qquad \lim_{n\to\infty}\|y_{n+1} - x_n\| = 0, \qquad \lim_{n\to\infty}\|y_{n+1} - y_n\| = 0.$$

By the definition of x_n and z_n in (3.1) we have

$$\begin{aligned}\|x_n - r\|^2 &= \|y_n + \beta_n (y_n - y_{n-1}) - r\|^2 \\ &\leq (\|y_n - r\| + \beta_n \|y_n - y_{n-1}\|)^2 .\end{aligned}$$

Thus,

$$\|x_n - r\|^2 \leq \|y_n - r\|^2 + 2\beta_n \|y_n - y_{n-1}\| \|y_n - r\| + \beta_n^2 \|y_n - y_{n-1}\|^2. \tag{3.5}$$

Also,

$$\begin{aligned}\|z_n - r\|^2 &= \|d_{n,0} x_n + \sum_{i=1}^{N} d_{n,i} T_i x_n - r\|^2 \\ &= \|d_{n,0}(x_n - r) + \sum_{i=1}^{N} (d_{n,i} T_i x_n - r)\|^2 \\ &= d_{n,0}\|x_n - r\|^2 + \sum_{i=1}^{N} d_{n,i}\|T_i x_n - r\|^2 - \sum_{0\leq i\leq 1}^{N} d_{n,0} d_{n,i} \|x_n - T_i x_n\| \\ &\leq d_{n,0}\|x_n - r\|^2 + \sum_{i=1}^{N} d_{n,i}\|x_n - r\|^2 - \sum_{0\leq i\leq 1}^{N} d_{n,0} d_{n,i} \|x_n - T_i x_n\| \\ &= \left(d_{n,0} + \sum_{i=1}^{N} d_{n,i}\right)\|x_n - r\|^2 - \sum_{0\leq i\leq 1}^{N} d_{n,0} d_{n,i} \|x_n - T_i x_n\|.\end{aligned}$$

We have

$$\|z_n - r\|^2 \leq \|x_n - r\|^2 - d_{n,0} d_{n,i} \|x_n - T_i x_n\|. \tag{3.6}$$

Again we obtain

$$\begin{aligned}\|z_n - r\|^2 &= \|d_{n,0} x_n + \sum_{i=1}^{N} d_{n,i} T_i x_n - r\|^2 \\ &\leq \left(d_{n,0}\|x_n - r\| + \sum_{i=1}^{N} d_{n,i}\|T_i x_n - r\|\right)^2 \\ &\leq \left(d_{n,0}\|x_n - r\| + \sum_{i=1}^{N} d_{n,i}\|x_n - r\|\right)^2 \\ &= \left(d_{n,0} + \sum_{i=1}^{N} d_{n,i}\right)\|x_n - r\|^2.\end{aligned}$$

Therefore,

$$\|z_n - r\|^2 \leq \|x_n - r\|^2. \tag{3.7}$$

Using (3.5), (3.6) and (3.7), we compute

$$\begin{aligned}
\|y_{n+1}-r\|^2 =&\|a_n f(y_n)+b_n x_n+c_n z_n-r\|^2\\
=&\|a_n(f(y_n)-r)+b_n(x_n-r)+c_n(z_n-r)\|^2\\
\le&a_n\|f(y_n)-r\|^2+b_n\|x_n-r\|^2+c_n\|z_n-r\|^2\\
\le&a_n\|f(y_n)-r\|^2+b_n\|x_n-r\|^2\\
&+c_n\left(\|x_n-r\|^2-d_{n,0}d_{n,i}\|x_n-T_ix_n\|\right)\\
=&a_n\|f(y_n)-r\|^2+b_n\|x_n-r\|^2\\
&+c_n\|x_n-r\|^2-c_nd_{n,0}d_{n,i}\|x_n-T_ix_n\|\\
=&a_n\|f(y_n)-r\|^2+(b_n+c_n)\|x_n-r\|^2-c_nd_{n,0}d_{n,i}\|x_n-T_ix_n\|\\
\le&a_n\|f(y_n)-r\|^2+(b_n+c_n)\Big[\|y_n-r\|^2+2\beta_n\|y_n-y_{n-1}\|\|y_n-r\|\\
&+\beta_n^2\|y_n-y_{n-1}\|^2\Big]-c_nd_{n,0}d_{n,i}\|x_n-T_ix_n\|\\
=&a_n\|f(y_n)-r\|^2+(b_n+c_n)\|y_n-r\|^2\\
&+2(b_n+c_n)\beta_n\|y_n-y_{n-1}\|\Big[\|y_n-r\|+\beta_n\|y_n-y_{n-1}\|\Big]\\
&-c_nd_{n,0}d_{n,i}\|x_n-T_ix_n\|\\
=&a_n\|f(y_n)-r\|^2+(1-a_n)\|y_n-r\|^2\\
&+2(b_n+c_n)\beta_n\|y_n-y_{n-1}\|\Big[\|y_n-r\|+\beta_n\|y_n-y_{n-1}\|\Big]\\
&-c_nd_{n,0}d_{n,i}\|x_n-T_ix_n\|\\
c_nd_{n,0}d_{n,i}\|x_n-T_ix_n\| \le&a_n\|f(y_n)-r\|^2+\|y_n-r\|^2-a_n\|y_n-r\|^2-\|y_{n+1}-r\|^2\\
&+2(b_n+c_n)\beta_n\|y_n-y_{n-1}\|\Big[\|y_n-r\|+\beta_n\|y_n-y_{n-1}\|\Big]\\
=&a_n\left(\|f(y_n)-r\|^2-\|y_n-r\|^2\right)-\left(\|y_{n+1}-r\|^2-\|y_n-r\|^2\right)\\
&+2(b_n+c_n)\beta_n\|y_n-y_{n-1}\|\Big[\|y_n-r\|+\beta_n\|y_n-y_{n-1}\|\Big].
\end{aligned}$$

By taking the limits of both side, also by conditions (1), (2), (3), (4) and the boundedness of $d_{n,0}, d_{n,i}$, we have

$$\lim_{n\to\infty}\|x_n-T_ix_n\|=0. \tag{3.8}$$

Also,

$$\begin{aligned}
\|x_n-y_n\| &= \|y_n+\beta_n(y_n-y_{n-1})-y_n\|\\
&\le \beta_n\|y_n-y_{n-1}\|.
\end{aligned}$$

We obtain

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.9}$$

Using (3.5) and (3.7), we compute

$$\begin{aligned}
\|y_{n+1}-r\|^2 =&\|a_n f(y_n)+b_n x_n+c_n z_n-r\|^2\\
=&\|a_n(f(y_n)-r)+b_n(x_n-r)+c_n(z_n-r)\|^2\\
\le&a_n\|f(y_n)-r\|^2+b_n\|x_n-r\|^2+c_n\|z_n-r\|^2-b_nc_n\|x_n-z_n\|^2\\
\le&a_n\|f(y_n)-r\|^2+b_n\|x_n-r\|^2+c_n\|x_n-r\|^2-b_nc_n\|x_n-z_n\|^2\\
=&a_n\|f(y_n)-r\|^2+(b_n+c_n)\|x_n-r\|^2-b_nc_n\|x_n-z_n\|^2\\
\le&a_n\|f(y_n)-r\|^2+(b_n+c_n)\Big[\|y_n-r\|^2+2\beta_n\|y_n-y_{n-1}\|\|y_n-r\|\\
&+\beta_n^2\|y_n-y_{n-1}\|^2\Big]-b_nc_n\|x_n-z_n\|^2\\
\le&a_n\|f(y_n)-r\|^2+(b_n+c_n)\|y_n-r\|^2\\
&+2(b_n+c_n)\beta_n\|y_n-y_{n-1}\|\Big[\|y_n-r\|+\beta_n\|y_n-y_{n-1}\|\Big]\\
&-b_nc_n\|x_n-z_n\|^2\\
=&a_n\|f(y_n)-r\|^2+(1-a_n)\|y_n-r\|^2\\
&+2(b_n+c_n)\beta_n\|y_n-y_{n-1}\|\Big[\|y_n-r\|+\beta_n\|y_n-y_{n-1}\|\Big]\\
&-b_nc_n\|x_n-z_n\|^2\\
b_nc_n\|x_n-z_n\|^2\le&a_n\|f(y_n)-r\|^2+\|y_n-r\|^2-a_n\|y_n-r\|^2-\|y_{n+1}-r\|^2\\
&+2(b_n+c_n)\beta_n\|y_n-y_{n-1}\|\Big[\|y_n-r\|+\beta_n\|y_n-y_{n-1}\|\Big]\\
=&a_n\left(\|f(y_n)-r\|^2-\|y_n-r\|^2\right)-\left(\|y_{n+1}-r\|^2-\|y_n-r\|^2\right)\\
&+2(b_n+c_n)\beta_n\|y_n-y_{n-1}\|\Big[\|y_n-r\|+\beta_n\|y_n-y_{n-1}\|\Big].
\end{aligned}$$

Taking the limits of both side, also by conditions (1), (2), (3) and (4), we have

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.10}$$

Also, we obtain

$$\begin{aligned}
\|z_n-y_n\|&=\|z_n-x_n+x_n-y_n\|\\
&\le\|z_n-x_n\|+\|x_n-y_n\|.
\end{aligned}$$

Using (3.9) and (3.10), we obtain

$$\lim_{n\to\infty}\|z_n-y_n\|=0. \tag{3.11}$$

Also,

$$\begin{aligned}
\|y_{n+1}-y_n\|&=\|a_n f(y_n)+b_n x_n+c_n z_n-y_n\|\\
&\le a_n\|f(y_n)-y_n\|+b_n\|x_n-y_n\|+c_n\|z_n-y_n\|.
\end{aligned}$$

Using condition (1), equations (3.9) and (3.11), we get

$$\lim_{n\to\infty}\|y_{n+1}-y_n\|=0. \tag{3.12}$$

Also, we have

$$\begin{aligned}
\|y_{n+1}-x_n\|&=\|a_n f(y_n)+b_n x_n+c_n z_n-x_n\|\\
&\le a_n\|f(y_n)-x_n\|+b_n\|x_n-x_n\|+c_n\|z_n-x_n\|.
\end{aligned}$$

Using condition (1) and equation (3.10), we obtain

$$\lim_{n\to\infty} \|y_{n+1} - x_n\| = 0. \tag{3.13}$$

Hence we compute

$$\begin{aligned}\|y_n - T_i y_n\| &= \|y_n - y_{n+1} + y_{n+1} - x_n + x_n - T_i x_n + T_i x_n - T_i y_n\| \\ &\leq \|y_n - y_{n+1}\| + \|y_{n+1} - x_n\| + \|x_n - T_i x_n\| + \|T_i x_n - T_i y_n\| \\ &\leq \|y_n - y_{n+1}\| + \|y_{n+1} - x_n\| + \|x_n - T_i x_n\| + \|x_n - y_n\|.\end{aligned}$$

Using (3.8), (3.9), (3.12) and (3.13), we have

$$\lim_{n\to\infty} \|y_n - T_i y_n\| = 0. \tag{3.14}$$

□

Lemma 3.4. *Let $\{y_n\}$ be a sequence defines in Lemma 3.1 and 3.2, where a_n, b_n, c_n, $\{d_{n,0}\}$ and $\{d_{n,i}\}$ are sequences in $(0,1)$ satisfying the conditions outlined above, then the sequence $\{y_n\}$ converges strongly to some $r \in \bigcap_{i=1}^{N} A(T_i)$ which characterizes the desired solution of the variational inequality problem*

$$\langle r - f(r), s - r\rangle \geq 0 \quad \forall \quad s \in \bigcap_{i=1}^{N} A(T_i). \tag{3.15}$$

Proof. Since our space is a Hilbert space and $\{y_n\}$ is bounded then we can find a subsequence $\{y_{n_p}\}$ of$\{y_n\}$ that converges weakly to s. By the definition of our mappings and from (3.14) with Lemma 2.1 we have $s \in \bigcap_{i=1}^{N} A(T_i)$ then (3.15) holds. Hence we have that
$\limsup_{n\to\infty}\langle f(r) - r, y_n - r\rangle = \lim_{p\to\infty}\langle f(r) - r, y_{n_p} - r\rangle = \langle f(r) - r, s - r\rangle \leq 0.$
Then,

$$\limsup_{n\to\infty}\langle f(r) - r, y_n - r\rangle \leq 0. \tag{3.16}$$

Using (3.2) and (3.3) we can obtained

$$\begin{aligned}\|y_{n+1} - r\|^2 =& \langle y_{n+1} - r, y_{n+1} - r\rangle \\ =& \langle a_n(f(y_n) - r) + b_n(x_n - r) + c_n(z_n - r), y_{n+1} - r\rangle \\ =& a_n\langle (f(y_n) - f(r) + f(r) - r), y_{n+1} - r\rangle \\ &+ b_n\langle x_n - r, y_{n+1} - r\rangle + c_n\langle z_n - r, y_{n+1} - r\rangle \\ =& a_n\langle f(y_n) - f(r), y_{n+1} - r\rangle + a_n\langle f(r) - r, y_{n+1} - r\rangle \\ &+ b_n\langle x_n - r, y_{n+1} - r\rangle + c_n\langle z_n - r, y_{n+1} - r\rangle \\ \leq& a_n L\|y_n - r\|\|y_{n+1} - r\| + b_n\|x_n - r\|\|y_{n+1} - r\| \\ &+ c_n\|z_n - r\|\|y_{n+1} - r\| + a_n\langle f(r) - r, y_{n+1} - r\rangle \\ \leq& a_n L\|y_n - r\|\|y_{n+1} - r\| + b_n\|x_n - r\|\|y_{n+1} - r\| \\ &+ c_n\|x_n - r\|\|y_{n+1} - r\| + a_n\langle f(r) - r, y_{n+1} - r\rangle \\ \leq& a_n L\|y_n - r\|\|y_{n+1} - r\| \\ &+ (b_n + c_n)(\|y_n - r\| + \beta_n\|y_n - y_{n-1}\|)\|y_{n+1} - r\| \\ &+ a_n\langle f(r) - r, y_{n+1} - r\rangle \\ =& a_n L\|y_n - r\|\|y_{n+1} - r\| + (b_n + c_n)\|y_n - r\|\|y_{n+1} - r\| \\ &+ (b_n + c_n)\beta_n\|y_n - y_{n-1}\|\|y_{n+1} - r\| + a_n\langle f(r) - r, y_{n+1} - r\rangle\end{aligned}$$

$$
\begin{aligned}
&= a_n L\|y_n - r\|\|y_{n+1} - r\| \\
&\quad + (b_n + c_n)\|y_n - r\|\|y_{n+1} - r\| \\
&\quad + (b_n + c_n)\beta_n\|y_n - y_{n-1}\|\|y_{n+1} - r\| \\
&\quad + a_n\langle f(r) - r, y_{n+1} - r\rangle \\
&= (a_n L + (b_n + c_n))\|y_n - r\|\|y_{n+1} - r\| \\
&\quad + (b_n + c_n)\beta_n\|y_n - y_{n-1}\|\|y_{n+1} - r\| \\
&\quad + a_n\langle f(r) - r, y_{n+1} - r\rangle \\
&\leq (a_n L + (b_n + c_n))\left[\frac{\|y_n - r\|^2 + \|y_{n+1} - r\|^2}{2}\right] \\
&\quad + (b_n + c_n)\beta_n\|y_n - y_{n-1}\|\|y_{n+1} - r\| \\
&\quad + a_n\langle f(r) - r, y_{n+1} - r\rangle \\
2\|y_{n+1} - r\|^2 &\leq (a_n L + (b_n + c_n))\left[\|y_n - r\|^2 + \|y_{n+1} - r\|^2\right] \\
&\quad + 2(b_n + c_n)\beta_n\|y_n - y_{n-1}\|\|y_{n+1} - r\| \\
&\quad + 2a_n\langle f(r) - r, y_{n+1} - r\rangle \\
\left[2 - (a_n L + (b_n + c_n))\right]\|y_{n+1} - r\|^2 &= (a_n L + (b_n + c_n))\|y_n - r\|^2 \\
&\quad + 2(b_n + c_n)\beta_n\|y_n - y_{n-1}\|\|y_{n+1} - r\| \\
&\quad + 2a_n\langle f(r) - r, y_{n+1} - r\rangle \\
&= \left[\frac{a_n L + (b_n + c_n)}{2 - (a_n L + (b_n + c_n))}\right]\|y_n - r\|^2 \\
&\quad + \frac{2(b_n + c_n)\beta_n}{2 - (a_n L + (b_n + c_n))}\|y_n - y_{n-1}\|\|y_{n+1} - r\| \\
&\quad + \frac{2a_n}{2 - (a_n L + (b_n + c_n))}\langle f(r) - r, y_{n+1} - r\rangle \\
&\leq \left[1 - \frac{2a_n(1 - L)}{2 - (a_n L + (b_n + c_n))}\right]\|y_n - r\|^2 \\
&\quad + \frac{a_n(1 - L)}{a_n(1 - L)}\left[\frac{2(b + c)\beta_n}{2 - (a_n L + (b_n + c_n))}\|y_n - y_{n-1}\|\|y_{n+1} - r\|\right] \\
&\quad + \frac{1 - L}{1 - L}\left[\frac{2a_n}{2 - (a_n L + (b_n + c_n))}\langle f(r) - r, y_{n+1} - r\rangle\right] \\
&= \left[1 - \frac{2a_n(1 - L)}{2 - (a_n L + (b_n + c_n))}\right]\|y_n - r\|^2 \\
&\quad + \frac{2a_n(1 - L)}{2 - (a_n L + b_n + c_n)}\left[\frac{\langle f(r) - r, y_{n+1} - r\rangle}{1 - L}\right] \\
&\quad + \frac{2a_n(1 - L)}{2 - (a_n L + b_n + c_n)}\left[\frac{(b + c)\beta_n}{a_n(1 - L)}\|y_n - y_{n-1}\|\|y_{n+1} - r\|\right].
\end{aligned}
$$

We see that by Lemma 2.2, we obtain

$$
\sum_{n=1}^{\infty}\sigma_n = \sum_{n=1}^{\infty}\frac{2a_n(1 - L)}{2 - (1 - a_n(1 - L))} > \sum_{n=1}^{\infty}\frac{2a_n(1 - L)}{2} = \sum_{n=1}^{\infty}a_n(1 - L) = \infty.
$$

By condition (1) and equation (3.16), we have
$\limsup_{n\to\infty} t_n = \limsup_{n\to\infty}\left[\frac{\langle f(r)-r, y_{n+1}-r\rangle}{1-L} + \frac{\beta_n}{a_n(1-L)}\|y_n - y_{n-1}\|\|y_{n+1} - r\|\right] \leq 0.$

Then by lemma 2.2, y_n converges strongly to $r \in \bigcap_{i=1}^{N} A(T_i)$. □

4. Conclusion

In this study, we developed a viscosity-inertial iterative scheme for approximating attractive points of widely more generalized hybrid mappings in Hilbert space. By incorporating inertial term into the viscosity framework, the proposed method achieves enhanced computational efficiency while maintaining strong convergence. Furthermore, we rigorously establish the strong convergence of the sequence generated by the scheme under relaxed assumptions.

Statements and Declarations

The authors declare that they have no conflict of interest.

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