



MODIFIED INERTIAL-TYPE KRASNOSEL'SKII-MANN ITERATIVE ALGORITHM INVOLVING TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper we prove strong convergence theorem for the modified Krasnoselskii-Mann iteration process with step-sizes involving total asymptotically nonexpansive mapping in the framework of Hilbert space. Our results generalize, unify, and improve several comparable results in literature.

Keywords. Krasnosel'skii-Mann algorithm, Step-size parameter, Fixed point problem, Hilbert space, Variational inequality problem, Total asymptotically nonexpansive mapping.

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1. INTRODUCTION

Let C be a closed, convex and nonempty subset of real Hilbert space H . A nonlinear mapping $T : C \rightarrow C$ is said to be

(i) a *contraction* if there exists a constant $\alpha \in [0, 1)$ such that

$$\|T(x) - T(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in C; \quad (1.1)$$

when $\alpha = 1$, then T is called *nonexpansive*;

(ii) *asymptotically nonexpansive* if there exists a real sequence $\{\mu_n\} \subseteq [0, +\infty)$ with $\lim_{n \rightarrow \infty} \mu_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \mu_n) \|x - y\|, \quad \forall x, y \in C, n \geq 1; \quad (1.2)$$

(iii) *asymptotically nonexpansive in the intermediate sense* if T is uniformly continuous and

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0; \quad (1.3)$$

if we let

$$\xi_n = \max \left\{ 0, \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \right\},$$

then $\lim_{n \rightarrow \infty} \xi_n = 0$, and the inequality becomes

$$\|T^n x - T^n y\| \leq \|x - y\| + \xi_n, \quad \forall x, y \in C, n \geq 1;$$

(iv) *generalized asymptotically nonexpansive* if there exist sequences $\{\mu_n\}, \{\xi_n\} \subseteq [0, +\infty)$ with $\mu_n \rightarrow 0, \xi_n \rightarrow 0$ as $n \rightarrow \infty$, such that

$$\|T^n x - T^n y\| \leq (1 + \mu_n) \|x - y\| + \xi_n, \quad \forall x, y \in C, n \geq 1; \quad (1.4)$$

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- (v) $(\{\mu_n\}, \{\xi_n\}, \zeta)$ -total asymptotically nonexpansive if there exist sequences $\{\mu_n\}, \{\xi_n\} \subseteq [0, +\infty)$ with $\mu_n \rightarrow 0, \xi_n \rightarrow 0$ as $n \rightarrow \infty$, and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\zeta(0) = 0$, such that

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \zeta(\|x - y\|) + \xi_n, \quad \forall x, y \in C, n \geq 1; \quad (1.5)$$

- (vi) uniformly L -Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad \forall x, y \in C, n \geq 1. \quad (1.6)$$

Note that If $\zeta(x) = x$, total asymptotically nonexpansive mappings coincide with generalized asymptotically nonexpansive mappings. In addition, if $\mu_n = 0$ for all $n \in \mathbb{N}$, then generalized asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings in the intermediate sense; if $\xi_n = 0$, then generalized asymptotically nonexpansive mappings coincide with asymptotically nonexpansive mappings, if $\mu_n = 0$ and $\xi_n = 0$, then we obtain nonexpansive mappings. These classes of mappings had been studied extensively by several authors (see e.g. [7, 8, 11, 15, 25]). In 2016, Alber et al. [1] introduced the concept of total asymptotically nonexpansive mappings that generalizes the family of mapping that are the extension of asymptotically nonexpansive mappings. The following examples show that total asymptotically nonexpansive mappings properly contain the asymptotically nonexpansive mappings.

Example 1.1. [13] Let $E = \mathbb{R} \times \ell_1$ be endowed with the norm

$$\|\cdot\|_E = |\cdot|_{\mathbb{R}} + \|\cdot\|_{\ell_1}.$$

Let K be a subset of E defined by

$$K := [0, 1] \times B,$$

where B is a closed unit ball of ℓ_1 . For all $u \in [0, 1]$ and $\bar{x} = (x_1, x_2, x_3, \dots) \in B$, define $T : K \rightarrow K$ by

$$T(u, \bar{x}) = \begin{cases} \left(\frac{1}{3}, \left(0, \frac{|x_1|^2}{3}, \frac{x_2}{3}, \frac{x_3}{3}, \frac{x_4}{3}, \dots \right) \right), & u \in \left[0, \frac{1}{3} \right], \\ \left(0, \left(0, \frac{|x_1|^2}{3}, \frac{x_2}{3}, \frac{x_3}{3}, \frac{x_4}{3}, \dots \right) \right), & u \in \left[\frac{1}{3}, 1 \right]. \end{cases} \quad (1.7)$$

We want to show that T given by (1.7) is not continuous and therefore cannot be asymptotically nonexpansive but satisfies condition (1.5).

Remark 1.2. every asymptotically nonexpansive mapping is uniformly Lipschitzian, so Lipschitz, and every Lipschitz mapping is continuous

Next, let $\{b_n\}_{n \geq 1}$ be a sequence of real numbers such that $b_1 = \frac{1}{3}$ and $\lim_{n \rightarrow \infty} b_n = 0$. Observe that for all $(u, \bar{x}), (v, \bar{y}) \in K$,

$$\|T(u, \bar{x}) - T(v, \bar{y})\|_E \leq |u - v| + b_1 + \frac{1}{3} \max\{|x_1| + |y_1|, 1\} \|\bar{x} - \bar{y}\|_{\ell_1}.$$

Moreover, for all $n \geq 2$ and all $(u, \bar{x}), (v, \bar{y}) \in K$,

$$T^n(u, \bar{x}) = \left(\frac{1}{3}, \left(0, 0, \dots, 0, \frac{|x_1|^2}{3^n}, \frac{x_2}{3^n}, \frac{x_3}{3^n}, \frac{x_4}{3^n}, \dots \right) \right),$$

and

$$\|T^n(u, \bar{x}) - T^n(v, \bar{y})\|_E \leq \frac{1}{3^n} \max\{|x_1| + |y_1|, 1\} \|\bar{x} - \bar{y}\|_{\ell_1}.$$

So, for all $n \geq 1$,

$$\|T^n(u, \bar{x}) - T^n(v, \bar{y})\|_E \leq |u - v| + \|\bar{x} - \bar{y}\|_{\ell_1} + \frac{2}{3^n} (|u - v| + \|\bar{x} - \bar{y}\|_{\ell_1}) + b_n. \quad (1.8)$$

Thus, with $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ defined by $\varphi(t) = 2t$, $\mu_n = \frac{1}{3^n}$ for all $n \geq 1$, and $\{b_n\}_{n \geq 1}$ any null sequence with $b_1 = \frac{1}{3}$, we obtain from (1.8) that

$$\|T^n(u, \bar{x}) - T^n(v, \bar{y})\|_E \leq \|(u, \bar{x}) - (v, \bar{y})\|_E + \mu_n \varphi(\|(u, \bar{x}) - (v, \bar{y})\|_E) + b_n.$$

Hence, the mapping T defined by (1.7) is total asymptotically nonexpansive but not asymptotically nonexpansive.

Example 1.3. [10] Let $X := \mathbb{R}$ and $C := [0, 2]$. Define $T : C \rightarrow C$ by

$$Tx = \begin{cases} 1, & x \in [0, 1], \\ \frac{1}{\sqrt{3}}\sqrt{4-x^2}, & x \in [1, 2]. \end{cases} \quad (1.9)$$

Note that $T^n x = 1$ for all $x \in C$ and $n \geq 2$, and $F(T) = \{1\}$. Clearly, T is both uniformly continuous and total asymptotically nonexpansive on C . We want to show that T given by (1.9) is not asymptotically nonexpansive. Hence, if $x \in [0, 1]$, then $\|x - 1\| = \|x - Tx\|$. Similarly, if $x \in [1, 2]$, then

$$\|x - 1\| = x - 1 \leq x - \frac{1}{\sqrt{3}}\sqrt{4-x^2} = \|x - Tx\|.$$

So, we get $d(x, F(T)) = \|x - 1\| \leq \|x - Tx\|$ for all $x \in C$. But T is not Lipschitzian. Indeed, suppose not, i.e., there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \text{for all } x, y \in C.$$

If we take $x = 2 - \frac{1}{3}(L+1)^2 > 1$ and $y = 2$, then

$$\frac{1}{\sqrt{3}}\sqrt{4-x^2} \leq L(2-x) \iff \frac{1}{3L^2} \leq \frac{2-x}{2+x}.$$

This is a contradiction.

Definition 1.4. Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $A : C \rightarrow H$ be a nonlinear mapping. The classical variational inequality is to find an $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \quad (1.10)$$

This is the classical variational inequality problem, and the set of its solutions is denoted by $SOL(A, C)$. Variational inequality problem in finite dimensional Hilbert spaces was first studied by Hartmann and Stampacchia [9] for modelling some non-linear elliptic differential-functional equations. Since then, several numerical methods have been developed which involves finding common solution to variational inequalities and related optimization problems where the operator is asymptotically nonexpansive in the framework of Hilbert space (see [17, 3, 24, 19, 4, 14, 20, 7, 6]). Furthermore, Ofoedu and Nnubia introduced the following algorithm:

Theorem 1.5. [13] Let K be a closed convex nonempty subset of a real Hilbert space H and let $T : K \rightarrow K$ be a uniformly continuous total asymptotically nonexpansive mapping with function $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying $\varphi(t) \leq M_0 t \forall t > M_1$ for some constants $M_0, M_1 > 0$. Suppose that $\text{Fix}(T) \neq \emptyset$ and let $\{x_n\}_{n \geq 1}$ be a sequence generated iteratively from $x_1 \in K$ by

$$\begin{cases} y_n = P_K[(1 - \alpha_n)x_n], \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T^n y_n, \quad n \geq 1, \end{cases} \quad (1.11)$$

where $\{\alpha_n\}_{n \geq 1}, \{\beta_n\}_{n \geq 1}$ are sequences in $(0, 1)$ satisfying the following conditions:

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \lim_{n \rightarrow \infty} \alpha_n = 0, \quad \lim_{n \rightarrow \infty} \alpha_n^{-1} \mu_n = 0 = \lim_{n \rightarrow \infty} \alpha_n^{-1} \eta_n,$$

and

$$0 < \zeta_0 < \beta_n < \varepsilon_0 < 1 \quad \forall n \geq 1 \quad (\text{for some } \zeta_0, \varepsilon_0 \in (0, 1)).$$

Then $\{x_n\}_{n \geq 1}$ converges strongly to $P_{\text{Fix}(T)}(0)$, which is a minimum norm fixed point of T .

Motivated by Theorem 1.5, we propose the following theorem.

Theorem 1.6. *Let C be a nonempty closed convex subset of the real Hilbert space H . Let $T : C \rightarrow C$ be a uniformly Lipschitzian and total asymptotically nonexpansive mapping with and let $(x_n)_{n \geq 0}$ be defined by*

$$\begin{cases} x_0, x_1 \in H. \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = (1 - a_n)T^n y_n, \\ x_{n+1} = \sigma_n z_n + b_n(T^n \sigma_n z_n - \sigma_n z_n) + \varepsilon_n, \quad \forall n \geq 1, \end{cases} \quad (1.12)$$

where $(\theta_n)_{n \geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, $(a_n)_{n \geq 1}, (b_n)_{n \geq 1} \subseteq [0, 1)$, $(\sigma_n)_{n \geq 1} \subseteq [0, 1]$, and $(\varepsilon_n)_{n \geq 1} \subseteq H$ are sequences such that $\lim_{n \rightarrow \infty} \|x_n - T^n x_n\| = 0$ and the $\text{Fix}(T) \neq \emptyset$. Assume that these conditions hold

$$(C1) \quad \sum_{n=0}^{\infty} a_n < \infty$$

$$(C2) \quad \limsup_{n \rightarrow \infty} \sigma_n = 1.$$

$$(C3) \quad \sum_{n=0}^{+\infty} \|\varepsilon_n\| < \infty.$$

$$(C4) \quad \text{There exists a constant } M_n > 0 \text{ for all } n \geq 0$$

If $\|T^n \sigma_n z_n - \sigma_n z_n\| \rightarrow 0$, then the sequence $(x_n)_{n \geq 0}$ converges strongly to $\hat{x} := P_{F(T)}(0)$.

It is our aim in this paper to prove strong convergence theorem for minimum norm fixed point of total asymptotically nonexpansive mapping. Our theorems generalize and extend the corresponding results of several results in literature.

2. PRELIMINARIES

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively, and let C be a nonempty, closed, and convex subset of H . Then we have the nearest point projection from H onto C , P_C , defined by

$$P_C(x) := \arg \min_{z \in C} \|x - z\|^2, \quad x \in H.$$

Thus, $P_C(x)$ is the only point in C that minimizes the objective $\|x - z\|^2$ over $z \in C$. Note that $P_C(x)$ is characterized as follows:

$$P_C(x) \in C \quad \text{and} \quad \langle x - P_C(x), y - P_C(x) \rangle \leq 0, \quad \forall y \in C.$$

In order to prove our results, we need the following lemmas.

Lemma 2.1. [18] *Let E be a real uniformly Banach space and $0 \leq p \leq t_n \leq q < 1$ for all positive integer $n \geq 1$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $\limsup_{n \rightarrow \infty} \|x_n\| < r$, $\limsup_{n \rightarrow \infty} \|y_n\| < r$ and $\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \geq 0$, then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$*

Lemma 2.2. [23] *Let α_n and t_n be two nonnegative sequences such that*

$$\alpha_{n+1} \leq \alpha_n + t_n, \quad \text{for all } n \geq 1.$$

If $\sum_{n=0}^{\infty} t_n < \infty$, then $\lim_{n \rightarrow \infty} \alpha_n$ exists.

Lemma 2.3. [5, 21, 22] Let C be a nonempty closed convex subset of H . Then for every $w \in H$ and $\hat{x} \in C$,

$$\hat{x} = P_C(w) \text{ if and only if } \langle w - \hat{x}, v - \hat{x} \rangle \leq 0, \forall v \in C.$$

Lemma 2.4. [21, 22] Let H be a real Hilbert space. Then, the following equality and inequality hold:

- i. $\|u + v\|^2 \leq \|u\|^2 + 2\langle u + v, v \rangle, \quad \forall u, v \in H.$
- ii. $\|\alpha u + (1 - \alpha)v\|^2 = \alpha\|u\|^2 + (1 - \alpha)\|v\|^2 - \alpha(1 - \alpha)\|u - v\|^2, \quad \forall \alpha \in \mathbb{R} \text{ and } u, v \in H.$

Lemma 2.5. [2] Let E be a reflexive Banach space with weakly continuous normalised duality mapping. Let K be a closed convex subset of E and let T be a uniformly continuous total asymptotically nonexpansive mapping from K into itself with bounded orbit. Then $(I - T)$ is demiclosed at zero.

Lemma 2.6. [12] Let $\{\rho_n\}_{n=1}^{\infty}, \{c_n\}_{n=1}^{\infty}, \{\epsilon_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^+ = [0, \infty)$, $b_n \in (0, 1)$ and $\{d_n\} \subseteq \mathbb{R}$ be sequences such that

$$\rho_{n+1} \leq [1 - b_n + c_n]\rho_n + d_n + \epsilon_n, \quad n \geq 1.$$

Let $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} \epsilon_n < \infty$. Then, we have the following:

- i. if $d_n \leq Mb_n$, for some $M > 0$, then $\{\rho_n\}$ is bounded
- ii. if $\lim_{n \rightarrow \infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n = \infty$ and $\limsup_{n \rightarrow \infty} \frac{d_n}{b_n} \leq 0$, then $\lim_{n \rightarrow \infty} d_n = 0$.

Lemma 2.7. [16] Let $\Phi : [0, \infty] \rightarrow [0, \infty)$ be a strictly increasing function with $\Phi(0) = 0$ and let $\{\rho_n\}, \{\lambda_n\}, \{\mu_n\}$ be nonnegative real sequences such that

$$\sum_{n=1}^{\infty} \lambda_n = \infty, \quad \lim_{n \rightarrow \infty} \mu_n = 0.$$

Suppose that

$$\rho_{n+1}^2 \leq \rho_n^2 - \lambda_n \Phi(\rho_{n+1}) + \lambda_n \mu_n, \quad n \geq 1. \quad (2.1)$$

Then

$$\lim_{n \rightarrow \infty} \rho_n = 0.$$

3. MAIN RESULT

Lemma 3.1. Let C be a nonempty closed convex subset of the real Hilbert space H . Let $T : C \rightarrow C$ be a uniformly Lipschitzian and total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$, and let $(x_n)_{n \geq 0}$ be defined by

$$\begin{cases} x_0, x_1 \in H. \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = (1 - a_n)T^n y_n, \\ x_{n+1} = \sigma_n z_n + b_n(T^n \sigma_n z_n - \sigma_n z_n) + \epsilon_n, \quad \forall n \geq 1, \end{cases} \quad (3.1)$$

Let $(\theta_n)_{n \geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$ such that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$. Assume that the conditions below hold.

Condition 3.2. (C1) $\sum_{n=0}^{\infty} a_n < \infty$

$$(C2) \limsup_{n \rightarrow \infty} \sigma_n = 1.$$

$$(C3) \sum_{n=0}^{+\infty} \|\varepsilon_n\| < \infty.$$

(C4) There exists a constant $M_n > 0$ for all $n \geq 0$

Then $(x_n)_{n \geq 0}$ is bounded.

Proof. Let $n \in \mathbb{N}$ and $u \in F(T)$. Then, consider

$$\begin{aligned} \|y_n - u\| &= \|x_n + \theta_n(x_n - x_{n-1}) - u\| \\ &\leq \|x_n - u\| + \theta_n \|x_n - x_{n-1}\|. \end{aligned} \quad (3.2)$$

Using (3.2), we have

$$\begin{aligned} \|z_n - u\| &= \|(1 - a_n)T^n y_n - u\| = \|(1 - a_n)(T^n y_n - u) - a_n u\| \\ &\leq (1 - a_n)\|T^n y_n - u\| + a_n \|u\| \\ &\leq (1 - a_n) [\|y_n - u\| + \mu_n \xi (\|y_n - u\|) + \varepsilon_n] + a_n \|u\|. \\ &\leq (1 - a_n) [\|x_n - u\| + \theta_n \|x_n - x_{n-1}\| + \mu_n \xi (\|x_n - u\| + \theta_n \|x_n - x_{n-1}\|) \\ &\quad + \varepsilon_n] + a_n \|u\|. \\ &\leq (1 - a_n)(1 + \mu_n \xi) [\|x_n - u\| + \theta_n \|x_n - x_{n-1}\|] + (1 - a_n)\|\varepsilon_n\| + a_n \|u\|. \end{aligned} \quad (3.3)$$

By using (3.3), it can be observed that

$$\begin{aligned} \|\sigma_n z_n - u\| &= \|\sigma_n(z_n - u) + (1 - \sigma_n)u\| \\ &\leq \sigma_n \|z_n - u\| + (1 - \sigma_n)\|u\| \\ &\leq \sigma_n [(1 - a_n)(1 + \mu_n \xi) [\|x_n - u\| + \theta_n \|x_n - x_{n-1}\|] + (1 - a_n)\|\varepsilon_n\| \\ &\quad + a_n \|u\|] + (1 - \sigma_n)\|u\| \\ &\leq \sigma_n(1 - a_n)(1 + \mu_n \xi) [\|x_n - u\| + \theta_n \|x_n - x_{n-1}\|] + \sigma_n(1 - a_n)\|\varepsilon_n\| \\ &\quad + \sigma_n a_n \|u\| + (1 - \sigma_n)\|u\| \\ &\leq \sigma_n(1 - a_n)(1 + \mu_n \xi) [\|x_n - u\| + \theta_n \|x_n - x_{n-1}\|] + \sigma_n(1 - a_n)\|\varepsilon_n\| \\ &\quad + (1 - \sigma_n(1 - a_n))\|u\| \end{aligned} \quad (3.4)$$

Using (3.4), for connecting, we will have

$$\begin{aligned} \|x_{n+1} - u\| &\leq \|\sigma_n z_n + b_n(T^n \sigma_n z_n - \sigma_n z_n) + \epsilon_n - u\| \\ &\leq (1 - b_n)\|\sigma_n z_n - u\| + b_n\|T^n \sigma_n z_n - u\| + \|\epsilon_n\| \\ &\leq (1 - b_n)\|\sigma_n z_n - u\| + b_n(\|\sigma_n z_n - u\| + \mu_n \xi \|\sigma_n z_n - u\| + \varepsilon_n) + \|\epsilon_n\| \\ &\leq (1 + b_n \mu_n \xi)\|\sigma_n z_n - u\| + b_n \varepsilon_n + \|\epsilon_n\| \\ &\leq (1 + b_n \mu_n \xi) [\sigma_n(1 - a_n)(1 + \mu_n \xi) [\|x_n - u\| + \theta_n \|x_n - x_{n-1}\|] \\ &\quad + \sigma_n(1 - a_n)\|\varepsilon_n\| + (1 - \sigma_n(1 - a_n))\|u\|] + b_n \varepsilon_n + \|\epsilon_n\| \\ &\leq (1 + b_n \mu_n \xi) \sigma_n(1 - a_n)(1 + \mu_n \xi) [\|x_n - u\| + \theta_n \|x_n - x_{n-1}\|] \\ &\quad + (1 + b_n \mu_n \xi) \sigma_n(1 - a_n)\|\varepsilon_n\| + (1 + b_n \mu_n \xi)(1 - \sigma_n(1 - a_n))\|u\| \\ &\quad + b_n \varepsilon_n + \|\epsilon_n\| \\ &\leq (1 + b_n \mu_n \xi) \sigma_n(1 - a_n)(1 + \mu_n \xi) \|x_n - u\| + (1 + b_n \mu_n \xi) \sigma_n(1 - a_n) \\ &\quad (1 + \mu_n \xi) \theta_n \|x_n - x_{n-1}\| + (1 + b_n \mu_n \xi) \sigma_n(1 - a_n) \|\varepsilon_n\| \\ &\quad + (1 + b_n \mu_n \xi)(1 - \sigma_n(1 - a_n))\|u\| + b_n \varepsilon_n + \|\epsilon_n\| \\ &\leq [1 - \beta_n + \mu_n] \|x_n - u\| + \delta_n + \lambda_n \end{aligned} \quad (3.5)$$

where

$$\begin{aligned}
\beta_n &= 1 - \sigma_n(1 - a_n) \\
\mu_n &= (1 + b_n\mu_n\xi)\sigma_n(1 - a_n)(1 + \mu_n\xi) \\
\delta_n &= (1 + b_n\mu_n\xi)(1 - \sigma_n(1 - a_n))\|u\| \\
\lambda_n &= (1 + b_n\mu_n\xi)\sigma_n(1 - a_n)(1 + \mu_n\xi)\theta_n\|x_n - x_{n-1}\| + b_n\varepsilon_n
\end{aligned}$$

Letting $\rho_n = \|x_n - u\|$ in (3.5), we obtain

$$\rho_{n+1} \leq [1 - \beta_n + \mu_n]\rho_n + \delta_n + \lambda_n.$$

Hence, by Lemma 2.6, the sequence $\|x_n - u\|$ is bounded. Consequently, $\{x_n\}_{n \geq 0}$ is bounded. \square

Lemma 3.3. *Let C be a nonempty closed convex subset of the real Hilbert space H . Let $T : C \rightarrow C$ be a uniformly Lipschitzian and total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$, and let $(x_n)_{n \geq 0}$ be defined by Algorithm (3.1). Let $(\theta_n)_{n \geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$ such that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty$. Assume that conditions 3.2 hold. Then $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow +\infty$.*

Proof. Let $n \in \mathbb{N}$ and $y \in F(T)$. Then, let us consider

$$\begin{aligned}
\|y_n - y_{n-1}\| &= \|x_n - x_{n-1} + \theta_n(x_n - x_{n-1}) - \theta_{n-1}(x_{n-1} - x_{n-2})\| \\
&\leq \|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\| + \theta_{n-1}\|x_{n-1} - x_{n-2}\|.
\end{aligned} \tag{3.6}$$

By using (3.6), we get that

$$\begin{aligned}
\|z_n - z_{n-1}\| &= \|(1 - a_n)T^n y_n - (1 - a_{n-1})T^n y_{n-1}\| \\
&= \|(1 - a_n)(T^n y_n - T^n y_{n-1}) - (a_n - a_{n-1})T^n y_{n-1}\| \\
&\leq (1 - a_n)\|(T^n y_n - T^n y_{n-1})\| + |(a_n - a_{n-1})|\|T^n y_{n-1}\| \\
&\leq (1 - a_n)[\|y_n - y_{n-1}\| + \mu_n\xi\|y_n - y_{n-1}\| + \varepsilon_n] + |(a_n - a_{n-1})|\|T^n y_{n-1}\|, \\
&\leq (1 - a_n)[(1 + \mu_n\xi)\|y_n - y_{n-1}\| + \varepsilon_n] + |(a_n - a_{n-1})|\|T^n y_{n-1}\|, \\
&\leq (1 - a_n)[(1 + \mu_n\xi)\|y_n - y_{n-1}\| + \varepsilon_n] + |(a_n - a_{n-1})|M_1, \\
&\leq (1 - a_n)[(1 + \mu_n\xi)(\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\| + \theta_{n-1}\|x_{n-1} - x_{n-2}\|) \\
&\quad + \varepsilon_n] + |(a_n - a_{n-1})|M_1 \\
&\leq (1 - a_n)(1 + \mu_n\xi)[(\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\| + \theta_{n-1}\|x_{n-1} - x_{n-2}\|) \\
&\quad + \varepsilon_n] + (1 - a_n)|(a_n - a_{n-1})|M_1
\end{aligned} \tag{3.7}$$

where $M_1 := \sup\{\|T^n y_{n-1}\| : n \in \mathbb{N}\}$. By connecting (3.7) with the inequality below, we obtain that

$$\begin{aligned}
\|\sigma_n z_n - \sigma_{n-1} z_{n-1}\| &= \|\sigma_n(z_n - z_{n-1}) + (\sigma_n - \sigma_{n-1})z_{n-1}\| \\
&\leq \sigma_n\|z_n - z_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|z_{n-1}\|, \\
&\leq \sigma_n[(1 - a_n)(1 + \mu_n\xi)[(\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\| \\
&\quad + \theta_{n-1}\|x_{n-1} - x_{n-2}\|) + \varepsilon_n] + (1 - a_n)|(a_n - a_{n-1})|M_1 \\
&\quad + |\sigma_n - \sigma_{n-1}|\|z_{n-1}\|, \\
&\leq \sigma_n[(1 - a_n)(1 + \mu_n\xi)[(\|x_n - x_{n-1}\| + \theta_n\|x_n - x_{n-1}\| \\
&\quad + \theta_{n-1}\|x_{n-1} - x_{n-2}\|) + \varepsilon_n] + (1 - a_n)|(a_n - a_{n-1})|M_1 \\
&\quad + |\sigma_n - \sigma_{n-1}|M_2,
\end{aligned} \tag{3.8}$$

where $M_2 := \sup\{\|z_{n-1}\| : n \in \mathbb{N}\}$. By employing (3.8), we have the following: $\|x_{n+1} - x_n\|$

$$\begin{aligned}
 &= \|(1 - b_n)\sigma_n z_n + b_n T^n \sigma_n z_n + \epsilon_n - ((1 - b_{n-1})\sigma_{n-1} z_{n-1} + b_{n-1} T^n \sigma_{n-1} z_{n-1} + \epsilon_{n-1})\| \\
 &= \|(1 - b_n)(\sigma_n z_n - \sigma_{n-1} z_{n-1}) + |b_n - b_{n-1}|(\sigma_{n-1} z_{n-1}) + b_n(T^n \sigma_n z_n - T^n \sigma_{n-1} z_{n-1}) \\
 &\quad + |b_n - b_{n-1}|(T^n \sigma_{n-1} z_{n-1}) + (\epsilon_n - \epsilon_{n-1})\| \\
 &\leq (1 - b_n)\|\sigma_n z_n - \sigma_{n-1} z_{n-1}\| + |b_n - b_{n-1}|\|\sigma_{n-1} z_{n-1}\| + b_n\|T^n \sigma_n z_n - T^n \sigma_{n-1} z_{n-1}\| \\
 &\quad + |b_n - b_{n-1}|\|T^n \sigma_{n-1} z_{n-1}\| + \|\epsilon_n - \epsilon_{n-1}\| \\
 &\leq (1 - b_n)\|\sigma_n z_n - \sigma_{n-1} z_{n-1}\| + |b_n - b_{n-1}|\|\sigma_{n-1} z_{n-1}\| + b_n[\|\sigma_n z_n - \sigma_{n-1} z_{n-1}\| \\
 &\quad + \mu_n \xi \|\sigma_n z_n - \sigma_{n-1} z_{n-1}\| + \varepsilon_n] + |b_n - b_{n-1}|\|T^n \sigma_{n-1} z_{n-1}\| + \|\epsilon_n - \epsilon_{n-1}\| \\
 &\leq (1 - b_n \mu_n \xi)\|\sigma_n z_n - \sigma_{n-1} z_{n-1}\| + b_n \varepsilon_n + |b_n - b_{n-1}|\|\sigma_{n-1} z_{n-1}\| \\
 &\quad + |b_n - b_{n-1}|\|T^n \sigma_{n-1} z_{n-1}\| + \|\epsilon_n - \epsilon_{n-1}\| \\
 &\leq (1 - b_n \mu_n \xi)[\sigma_n [(1 - a_n)(1 + \mu_n \xi)(\|x_n - x_{n-1}\| + \theta_n \|x_n - x_{n-1}\| \\
 &\quad + \theta_{n-1} \|x_{n-1} - x_{n-2}\|) + \varepsilon_n] + (1 - a_n)|(a_n - a_{n-1})|M_1] \\
 &\quad + |\sigma_n - \sigma_{n-1}|M_2 + b_n \varepsilon_n + |b_n - b_{n-1}|\|\sigma_{n-1} z_{n-1}\| \\
 &\quad + |b_n - b_{n-1}|\|T^n \sigma_{n-1} z_{n-1}\| + \|\epsilon_n - \epsilon_{n-1}\| \\
 &\leq (1 - b_n \mu_n \xi)\sigma_n(1 - a_n)(1 + \mu_n \xi)\|x_n - x_{n-1}\| + (1 - b_n \mu_n \xi)\sigma_n(1 - a_n)(1 + \mu_n \xi)\theta_n \|x_n - x_{n-1}\| \\
 &\quad + (1 - b_n \mu_n \xi)\sigma_n(1 - a_n)(1 + \mu_n \xi)\theta_{n-1} \|x_{n-1} - x_{n-2}\| + (1 - b_n \mu_n \xi)\sigma_n(1 - a_n)(1 + \mu_n \xi)\varepsilon_n \\
 &\quad + \sigma_n(1 - a_n)|(a_n - a_{n-1})|M_1 + (1 - b_n \mu_n \xi)|\sigma_n - \sigma_{n-1}|M_2 + b_n \varepsilon_n \\
 &\quad + |b_n - b_{n-1}|M_3 + \|\epsilon_n - \epsilon_{n-1}\| \\
 &= [1 - \beta_n + \mu_n]\|x_n - x_{n-1}\| + \delta_n + \lambda_n
 \end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
 \beta_n &= 1 - \sigma_n(1 - a_n) \\
 \mu_n &= (1 - b_n \mu_n \xi)\sigma_n(1 - a_n)(1 + \mu_n \xi) \\
 \delta_n &= (1 - b_n \mu_n \xi)\sigma_n(1 - a_n)(1 + \mu_n \xi)\theta_n \|x_n - x_{n-1}\| \\
 &\quad + (1 - b_n \mu_n \xi)\sigma_n(1 - a_n)(1 + \mu_n \xi)[\theta_{n-1} \|x_{n-1} - x_{n-2}\| + \varepsilon_n] \\
 &\quad + \sigma_n(1 - a_n)|(a_n - a_{n-1})|M_1 + (1 - b_n \mu_n \xi)|\sigma_n - \sigma_{n-1}|M_2 \\
 \lambda_n &= b_n \varepsilon_n + |b_n - b_{n-1}|M_3 + \|\epsilon_n - \epsilon_{n-1}\|
 \end{aligned}$$

where $M_3 := \sup\{\|T^n \sigma_{n-1} z_{n-1}\| + \|\sigma_{n-1} z_{n-1}\| : n \in \mathbb{N}\}$. Letting $\rho_n = \|x_n - x_{n-1}\|$ in (3.9), we obtain

$$\rho_{n+1} \leq [1 - \beta_n + \mu_n]\rho_n + \delta_n + \lambda_n.$$

Hence, by Lemma 2.6, the sequence $\|x_n - x_{n-1}\|$ is bounded. Consequently, we conclude that

$$\|x_{n+1} - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

□

Theorem 3.4. *Let C be a nonempty closed convex subset of the real Hilbert space H . Let $T : C \rightarrow C$ be a uniformly Lipschitzian and total asymptotically nonexpansive mapping with and let $(x_n)_{n \geq 0}$ be defined by*

$$\begin{cases} x_0, x_1 \in H. \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = (1 - a_n)T^n y_n, \\ x_{n+1} = \sigma_n z_n + b_n(T^n \sigma_n z_n - \sigma_n z_n) + \epsilon_n, \quad \forall n \geq 1, \end{cases} \tag{3.10}$$

where $(\theta_n)_{n \geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, $(a_n)_{n \geq 1}, (b_n)_{n \geq 1} \subseteq [0, 1)$, $(\sigma_n)_{n \geq 1} \subseteq [0, 1]$, and $(\varepsilon_n)_{n \geq 1} \subseteq H$ are sequences such that $\lim_{n \rightarrow \infty} \|x_n - T^n x_n\| = 0$ and the $\text{Fix}(T) \neq \emptyset$. Assume that these conditions hold

$$(C1) \sum_{n=0}^{\infty} a_n < \infty$$

$$(C2) \limsup_{n \rightarrow \infty} \sigma_n = 1.$$

$$(C3) \sum_{n=0}^{+\infty} \|\varepsilon_n\| < \infty.$$

$$(C4) \text{ There exists a constant } M_n > 0 \text{ for all } n \geq 0$$

If $\|T^n \sigma_n z_n - \sigma_n z_n\| \rightarrow 0$, then the sequence $(x_n)_{n \geq 0}$ converges strongly to $\hat{x} := P_{F(T)}(0)$.

Proof. From Lemma 3.1, we have $(x_n)_{n \geq 0}$ is bounded. Since $F(T) \neq \emptyset$, $y_n = x_n + \theta_n(x_n - x_{n-1})$ and $z_n = (1 - a_n)T^n y_n$, so $(y_n)_{n \geq 0}$ are also bounded. Let $\hat{x} := P_{F(T)}(0)$. Then $\hat{x} \in F(T)$. By using Lemma 2.4, we get that

$$\begin{aligned} \|y_n - \hat{x}\|^2 &= \|x_n - \hat{x} + \theta_n(x_n - x_{n-1})\|^2 \leq \|x_n - \hat{x}\|^2 + 2\theta_n \langle x_n - \hat{x}, x_n - x_{n-1} \rangle \\ &\leq \|x_n - \hat{x}\|^2 + \theta_n \|(x_n - x_{n-1})\|^2 L_1 \end{aligned} \quad (3.11)$$

where $L_1 := \sup\{2\|x_n - \hat{x}\| : n \in \mathbb{N}\}$. Therefore, using equation (3.11), we get that

$$\begin{aligned} \|z_n - \hat{x}\|^2 &= \|(1 - a_n)T^n y_n - \hat{x}\|^2 = \|T^n y_n - \hat{x} - a_n T^n y_n\|^2 \\ &= \left\| (1 - a_n) \left(\frac{1}{(1 - a_n)} (T^n y_n - \hat{x}) + \frac{-a_n}{(1 - a_n)} T^n y_n \right) \right\|^2 \\ &= (1 - a_n)^2 \left\| \frac{1}{(1 - a_n)} (T^n y_n - \hat{x}) + \frac{-a_n}{(1 - a_n)} T^n y_n \right\|^2 \\ &= (1 - a_n)^2 \left(\frac{1}{(1 - a_n)^2} \|(T^n y_n - \hat{x})\|^2 + \frac{-a_n}{(1 - a_n)} \|T^n y_n\|^2 + \frac{a_n}{(1 - a_n)^2} \|\hat{x}\|^2 \right) \\ &= (1 - a_n) \|T^n y_n - \hat{x}\|^2 - a_n(1 - a_n) \|T^n y_n\|^2 + a_n \|\hat{x}\|^2 \\ &= (1 - a_n) [\|y_n - \hat{x}\|^2 + \mu_n \xi \|y_n - \hat{x}\|^2 + \varepsilon_n] - a_n(1 - a_n) \|T^n y_n\|^2 + a_n \|\hat{x}\|^2 \\ &\leq (1 - a_n) [(1 + \mu_n \xi) (\|x_n - \hat{x}\|^2 + \theta_n \|(x_n - x_{n-1})\|^2 L_1) + \varepsilon_n] \\ &\quad - a_n(1 - a_n) \|T^n y_n\|^2 + a_n \|\hat{x}\|^2 \\ &\leq (1 - a_n)(1 + \mu_n \xi) \|x_n - \hat{x}\|^2 + (1 - a_n)(1 + \mu_n \xi) \theta_n \|(x_n - x_{n-1})\|^2 L_1 \\ &\quad + (1 - a_n) \varepsilon_n - a_n(1 - a_n) \|T^n y_n\|^2 + a_n \|\hat{x}\|^2 \end{aligned} \quad (3.12)$$

Employing (3.12), we obtained that

$$\begin{aligned} \|\sigma_n z_n - \hat{x}\|^2 &= \|\sigma_n(z_n - \hat{x}) + (\sigma_n - 1)\hat{x}\|^2 \\ &= \sigma_n^2 \|z_n - \hat{x}\|^2 + 2\sigma_n(1 - \sigma_n) \langle -\hat{x}, z_n - \hat{x} \rangle + (1 - \sigma_n)^2 \|\hat{x}\|^2 \\ &\leq \sigma_n^2 ((1 - a_n)(1 + \mu_n \xi) \|x_n - \hat{x}\|^2 + (1 - a_n)(1 + \mu_n \xi) \theta_n \|(x_n - x_{n-1})\|^2 L_1 \\ &\quad + (1 - a_n) \varepsilon_n - a_n(1 - a_n) \|T^n y_n\|^2 + a_n \|\hat{x}\|^2) + 2\sigma_n(1 - \sigma_n) \langle -\hat{x}, z_n - \hat{x} \rangle \\ &\quad + (1 - \sigma_n)^2 \|\hat{x}\|^2 \\ &\leq \sigma_n^2 (1 - a_n)(1 + \mu_n \xi) \|x_n - \hat{x}\|^2 + \sigma_n^2 (1 - a_n)(1 + \mu_n \xi) \theta_n \|(x_n - x_{n-1})\|^2 L_1 \\ &\quad + \sigma_n^2 (1 - a_n) \varepsilon_n - \sigma_n^2 a_n (1 - a_n) \|T^n y_n\|^2 + \sigma_n^2 a_n \|\hat{x}\|^2 + 2\sigma_n(1 - \sigma_n) \\ &\quad \langle -\hat{x}, z_n - \hat{x} \rangle + (1 - \sigma_n)^2 \|\hat{x}\|^2 \end{aligned} \quad (3.13)$$

By using (3.13), it can be observed that

$$\begin{aligned}
 \|x_{n+1} - \hat{x}\|^2 &= \|\sigma_n z_n + b_n(T^n \sigma_n z_n - \sigma_n z_n) + \varepsilon_n - \hat{x}\|^2 \\
 &= \|(1 - b_n)(\sigma_n z_n - \hat{x}) + b_n(T^n \sigma_n z_n - \hat{x}) + \varepsilon_n\|^2 \\
 &\leq \|(1 - b_n)(\sigma_n z_n - \hat{x}) + b_n(T^n \sigma_n z_n - \hat{x})\|^2 + 2\langle x_{n+1} - \hat{x}, \varepsilon_n \rangle \\
 &\leq (1 - b_n)\|\sigma_n z_n - \hat{x}\|^2 + b_n\|T^n \sigma_n z_n - \hat{x}\|^2 + \|\varepsilon_n\|L_2 \\
 &\leq (1 - b_n)\|\sigma_n z_n - \hat{x}\|^2 + b_n[\|\sigma_n z_n - \hat{x}\|^2 + \mu_n \xi \|\sigma_n z_n - \hat{x}\|^2 + \varepsilon_n] + \|\varepsilon_n\|L_2 \\
 &\leq (1 - b_n)\|\sigma_n z_n - \hat{x}\|^2 + b_n[(1 + \mu_n \xi)\|\sigma_n z_n - \hat{x}\|^2 + \varepsilon_n] + \|\varepsilon_n\|L_2 \\
 &\leq (1 + b_n \mu_n \xi)\|\sigma_n z_n - \hat{x}\|^2 + b_n \varepsilon_n + \|\varepsilon_n\|L_2 \\
 &\leq (1 + b_n \mu_n \xi)\sigma_n^2(1 - a_n)(1 + \mu_n \xi)\|x_n - \hat{x}\|^2 \\
 &\quad + (1 + b_n \mu_n \xi)\sigma_n^2(1 - a_n)(1 + \mu_n \xi)\theta_n\|(x_n - x_{n-1})\|^2 L_1 \\
 &\quad + (1 + b_n \mu_n \xi)\sigma_n^2(1 - a_n)\varepsilon_n - (1 + b_n \mu_n \xi)\sigma_n^2 a_n(1 - a_n)\|T^n y_n\|^2 \\
 &\quad + (1 + b_n \mu_n \xi)\sigma_n^2 a_n \|\hat{x}\|^2 + 2(1 + b_n \mu_n \xi)\sigma_n(1 - \sigma_n)\langle -\hat{x}, z_n - \hat{x} \rangle \\
 &\quad + (1 + b_n \mu_n \xi)(1 - \sigma_n)^2 \|\hat{x}\|^2 + b_n \varepsilon_n + \|\varepsilon_n\|L_2 \\
 &\leq [1 - \beta_n + \mu_n]\rho_n^2 + \delta_n
 \end{aligned} \tag{3.14}$$

where $L_2 := \sup\{2\|x_{n+1} - \hat{x}\| : n \in \mathbb{N}\}$ and

$$\begin{aligned}
 \beta_n &= 1 - \sigma_n^2(1 - a_n) \\
 \mu_n &= (1 + b_n \mu_n \xi)\sigma_n^2(1 - a_n)(1 + \mu_n \xi) \\
 \delta_n &= (1 + b_n \mu_n \xi)\sigma_n^2(1 - a_n)(1 + \mu_n \xi)\theta_n\|(x_n - x_{n-1})\|^2 L_1 \\
 &\quad + (1 + b_n \mu_n \xi)\sigma_n^2(1 - a_n)\varepsilon_n - (1 + b_n \mu_n \xi)\sigma_n^2 a_n(1 - a_n)\|T^n y_n\|^2 \\
 &\quad + (1 + b_n \mu_n \xi)\sigma_n^2 a_n \|\hat{x}\|^2 + 2(1 + b_n \mu_n \xi)\sigma_n(1 - \sigma_n)\langle -\hat{x}, z_n - \hat{x} \rangle \\
 &\quad + (1 + b_n \mu_n \xi)(1 - \sigma_n)^2 \|\hat{x}\|^2 + b_n \varepsilon_n + \|\varepsilon_n\|L_2
 \end{aligned}$$

Letting $\rho_n = \|x_n - \hat{x}\|$ in (3.14), we obtain

$$\rho_{n+1}^2 \leq [1 - \beta_n + \mu_n]\rho_n^2 + \delta_n.$$

Hence, by Lemma 2.7 the sequence $\|x_n - \hat{x}\|$ is bounded. Next, we will prove that $\|T^n \sigma_n z_n - \sigma_n z_n\| \rightarrow 0$ as $n \rightarrow \infty$. We observe that

$$\begin{aligned}
 \|T^n x_{n+1} - z_n\| &= \|T^n x_{n+1} - (1 - a_n)T^n y_n\| = \|T^n x_{n+1} - T^n y_n + a_n T^n y_n\| \\
 &\leq \|T^n x_{n+1} - T^n y_n\| + a_n \|T^n y_n\| \\
 &\leq \|x_{n+1} - y_n\| + \mu_n \xi \|x_{n+1} - y_n\| + \varepsilon_n + a_n L_3 \\
 &\leq (1 + \mu_n \xi)\|x_{n+1} - y_n\| + \varepsilon_n + a_n L_3 \\
 &\leq (1 + \mu_n \xi)\|x_{n+1} - x_n - \theta_n(x_n - x_{n-1})\| + \varepsilon_n + a_n L_3 \\
 &\leq (1 + \mu_n \xi)\|x_{n+1} - x_n\| + (1 + \mu_n \xi)\theta_n\|(x_n - x_{n-1})\| + \varepsilon_n + a_n L_3
 \end{aligned} \tag{3.15}$$

where $L_3 := \sup\{\|T^n y_n\| : n \in \mathbb{N}\}$. By using (3.15), we get that

$$\begin{aligned}
\|T^n \sigma_n z_n - \sigma_n z_n\| &= \|T^n \sigma_n z_n - T^n x_{n+1} + T^n x_{n+1} - \sigma_n z_n\| \\
&\leq \|T^n \sigma_n z_n - T^n x_{n+1}\| + \|T^n x_{n+1} - \sigma_n z_n\| \\
&\leq (1 + \mu_n \xi) \|\sigma_n z_n - x_{n+1}\| + \varepsilon_n + \|(1 - \sigma_n) T^n x_{n+1} + \sigma_n (T^n x_{n+1} - z_n)\| \\
&\leq (1 + \mu_n \xi) \|\sigma_n z_n - (\sigma_n z_n + b_n (T^n \sigma_n z_n - \sigma_n z_n) + \epsilon_n)\| \\
&\quad + \|(1 - \sigma_n) T^n x_{n+1} + \sigma_n (T^n x_{n+1} - z_n)\| + \varepsilon_n \\
&\leq (1 + \mu_n \xi) b_n \|T^n \sigma_n z_n - \sigma_n z_n\| + (1 + \mu_n \xi) \|\epsilon_n\| + (1 - \sigma_n) L_4 \\
&\quad + \sigma_n \|T^n x_{n+1} - z_n\| + \varepsilon_n \\
&\leq (1 + \mu_n \xi) b_n \|T^n \sigma_n z_n - \sigma_n z_n\| + (1 + \mu_n \xi) \|\epsilon_n\| + (1 - \sigma_n) L_4 \\
&\quad + \sigma_n [(1 + \mu_n \xi) \|x_{n+1} - x_n\| + (1 + \mu_n \xi) \theta_n \|(x_n - x_{n-1})\| \\
&\quad + \varepsilon_n + a_n L_3] + \varepsilon_n
\end{aligned} \tag{3.16}$$

where $L_4 := \sup\{\|T^n x_{n+1}\| : n \in \mathbb{N}\}$. It follows from (3.16), conditions (C1), (C3), (C4) and Lemma 3.3 that

$$\begin{aligned}
\|T^n \sigma_n z_n - \sigma_n z_n\| &\leq \frac{1}{(1 + \mu_n \xi) b_n - 1} [(1 + \mu_n \xi) \|\epsilon_n\| + (1 - \sigma_n) L_4 \\
&\quad + \sigma_n [(1 + \mu_n \xi) \|x_{n+1} - x_n\| + (1 + \mu_n \xi) \theta_n \|(x_n - x_{n-1})\| \\
&\quad + \varepsilon_n + a_n L_3] + \varepsilon_n \\
&\rightarrow 0 \text{ as } n \rightarrow \infty
\end{aligned} \tag{3.17}$$

From (3.17) we conclude that

$$\lim_{n \rightarrow \infty} \|T \sigma_n z_n - \sigma_n z_n\| = 0. \tag{3.18}$$

Next, we expect that the sequence $(x_n)_{n \geq 0}$ converge strongly to \hat{x} which is enough to show that

$$\limsup_{n \rightarrow \infty} \langle -\hat{x}, z_n - \hat{x} \rangle \leq 0 \tag{3.19}$$

Let us assume on the contrary that (3.19) does not hold. Then, there exists a real number $r > 0$ and a subsequence $(z_{n_m})_{m \geq 1} \in (z_n)_{n \geq 1}$ such that

$$\langle -\hat{x}, z_{n_m} - \hat{x} \rangle \geq r > 0, \quad \forall m \geq 1.$$

The boundedness of $(z_{n_m})_{m \geq 1}$ implies that there is a subsequence $(z_{n_{m_l}})_{l \geq 1}$ of $(z_{n_m})_{m \geq 1}$ such that $z_{n_{m_l}} \rightharpoonup z \in H$ as $l \rightarrow +\infty$. Therefore,

$$0 < r \leq \lim_{l \rightarrow \infty} \langle -\hat{x}, z_{n_{m_l}} - \hat{x} \rangle = \langle -\hat{x}, z - \hat{x} \rangle \tag{3.20}$$

since $\lim_{n \rightarrow \infty} \sigma_n = 1$, so we get that

$$\sigma_{n_{m_l}} z_{n_{m_l}} \rightharpoonup z \text{ as } n \rightarrow \infty. \tag{3.21}$$

By (3.18), (3.21) and Lemma 2.5, it implies that $z \in F(T)$. Due to Lemma 2.3, the inequality

$$\langle -\hat{x}, z - \hat{x} \rangle = \langle 0 - \hat{x}, z - \hat{x} \rangle \leq 0$$

is valid which causes a contradiction with (3.20). Therefore, it leads to the conclusion (3.19) is true. And then, condition (C2) ensures that

$$\limsup_{n \rightarrow \infty} (2\sigma_n \langle -\hat{x}, z_n - \hat{x} \rangle + (1 - \sigma_n) \|\hat{x}\|^2) \leq 0.$$

Finally (3.14) and Lemma 2.2 (2) give us the desired result, that is

$$\lim_{n \rightarrow \infty} x_n = \hat{x}.$$

The proof is complete. \square

Remark 3.5. Let $\{\xi_n\}_{n \geq 1} \subseteq [0, +\infty)$ be the sequence such that $\sum_{n=1}^{\infty} \xi_n < +\infty$. Then, we define

$$\tilde{\theta}_n = \begin{cases} \min \left\{ \theta, \frac{\xi_n}{\|x_n - x_{n-1}\|} \right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise,} \end{cases}$$

where $(x_n)_{n \geq 0}$ and θ are taken from Theorem 3.4. Next, if $(\theta_n)_{n \geq 1}$ is chosen from $[0, \tilde{\theta}_n]$ for all $n \in \mathbb{N}$, then it is not hard to verify that

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < +\infty.$$

Corollary 3.6. *Let C be a nonempty closed convex subset of the real Hilbert space H . Let $T : C \rightarrow C$ be a uniformly Lipschitzian and total asymptotically nonexpansive mapping and let $(x_n)_{n \geq 0}$ be defined by*

$$\begin{cases} x_0, x_1 \in H. \\ y_n = s_n x_n + (1 - s_n) x_{n-1}, \\ z_n = (1 - a_n) T^n y_n, \\ x_{n+1} = \sigma_n z_n + b_n (T^n \sigma_n z_n - \sigma_n z_n) + \varepsilon_n, \quad \forall n \geq 1, \end{cases} \quad (3.22)$$

where $(s_n)_{n \geq 1} \subseteq (0, 1)$ and $\lim_{n \rightarrow 1} s_n = s \in (0, 1)$, $(a_n)_{n \geq 1}, (b_n)_{n \geq 1} \subseteq [0, 1)$, $(\sigma_n)_{n \geq 1} \subseteq [0, 1]$, and $(\varepsilon_n)_{n \geq 1} \subseteq H$ are sequences such that $\lim_{n \rightarrow \infty} \|x_n - T^n x_n\| = 0$ and the $\text{Fix}(T) \neq \emptyset$. Assume that these conditions in Theorem 3.4 holds. If $\|T^n \sigma_n z_n - \sigma_n z_n\| \rightarrow 0$, then the sequence $(x_n)_{n \geq 0}$ converges strongly to $\hat{x} := P_{F(T)}(0)$.

Proof. Take $\theta_n = s_n - 1$ in Theorem 3.4. Let $(s_n)_{n \geq 1} \subseteq (0, 1)$ and $\lim_{n \rightarrow 1} s_n = s \in (0, 1)$ with condition (3.2) satisfied, hence the conclusion of the proof follows from Theorem 3.4, this completes the proof. \square

Remark 3.7. Corollary 3.6 studied the strongly convergence theorem without the monotonic increase of the sequence $\{s_n\}$, and so improves and extends the main results in [3].

Corollary 3.8. *Let C be a nonempty closed convex subset of the real Hilbert space H . Let $T : C \rightarrow C$ be a uniformly Lipschitzian and total asymptotically nonexpansive mapping and let $(x_n)_{n \geq 0}$ be defined by*

$$\begin{cases} x_0, x_1 \in H. \\ y_n = \frac{1}{2}(x_n + x_{n-1}), \\ z_n = (1 - a_n) T^n y_n, \\ x_{n+1} = \sigma_n z_n + b_n (T^n \sigma_n z_n - \sigma_n z_n) + \varepsilon_n, \quad \forall n \geq 1, \end{cases} \quad (3.23)$$

where $(a_n)_{n \geq 1}, (b_n)_{n \geq 1} \subseteq [0, 1)$, $(\sigma_n)_{n \geq 1} \subseteq [0, 1]$, and $(\varepsilon_n)_{n \geq 1} \subseteq H$ are sequences such that $\lim_{n \rightarrow \infty} \|x_n - T^n x_n\| = 0$ and the $\text{Fix}(T) \neq \emptyset$. Assume that these conditions in Theorem 3.4 holds. If $\|T^n \sigma_n z_n - \sigma_n z_n\| \rightarrow 0$, then the sequence $(x_n)_{n \geq 0}$ converges strongly to $\hat{x} := P_{F(T)}(0)$.

Proof. Take $s_n = \frac{1}{2}$ in Corollary 3.6. Let $(s_n)_{n \geq 1} \subseteq (0, 1)$ and $\lim_{n \rightarrow 1} s_n = s \in (0, 1)$ with condition (3.2) satisfied, hence the conclusion of the proof follows from Theorem 3.4, this completes the proof. \square

Remark 3.9. Corollary 3.6 and 3.8 are new consequences.

4. APPLICATION TO VARIATIONAL INEQUALITY PROBLEM

Assume that C is a nonempty closed convex subset of a real Hilbert space H . Let $A : H \rightarrow H$ be a single-valued monotone operator such that $C \subset \text{dom}(A)$. Next we consider the following variational inequality (VI):

$$\langle Ax_0, x - x_0 \rangle \geq 0, \quad x \in C. \quad (4.1)$$

Notice that VI (4.1) is equivalent to the fixed point problem, for any $\lambda > 0$,

$$P_C(I - \lambda A)x_0 = x_0. \quad (4.2)$$

Definition 4.1. A nonlinear mapping $A : H \rightarrow H$ is L -Lipschitzian for some $L > 0$, if

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in H.$$

Definition 4.2. A nonlinear mapping $A : H \rightarrow H$ is η -inverse-strongly monotone, for some $\eta > 0$, if

$$\langle Ax - Ay, x - y \rangle \geq \eta\|Ax - Ay\|^2, \quad \forall x, y \in H. \quad (4.3)$$

If A is Lipschitzian and η -inverse-strongly monotone, it is well known that the operator $T = P_C(I - \lambda A)$ is nonexpansive provided $0 < \lambda < 2\eta$. Thus, we can get the following theorem.

Theorem 4.3. Assume that C is a nonempty closed convex subset of the real Hilbert space H . Let $A : H \rightarrow H$ be an L -Lipschitzian and η -inverse-strongly monotone mapping and $f : C \rightarrow C$ a contraction with coefficient $\alpha \in [0, 1)$. Assume VI (4.1) is solvable. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0, x_1 \in H. \\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = (1 - a_n)P_C(1 - \mu A)y_n, \\ x_{n+1} = \sigma_n z_n f(x_n) + b_n(P_C(1 - \lambda A)\sigma_n z_n - \sigma_n z_n) + \varepsilon_n, \quad \forall n \geq 1, \end{cases} \quad (4.4)$$

where $0 < \lambda < 2\eta$ and $(\theta_n)_{n \geq 1} \subseteq [0, \theta]$ with $\theta \in [0, 1)$, $(a_n)_{n \geq 1}, (b_n)_{n \geq 1} \subseteq [0, 1)$, $(\sigma_n)_{n \geq 1} \subseteq [0, 1]$, and $(\varepsilon_n)_{n \geq 1} \subseteq H$ are sequences such that $\lim_{n \rightarrow \infty} \|x_n - T^n x_n\| = 0$. Assume that conditions in Theorem 3.4 holds. Then, the sequence $(x_n)_{n \geq 0}$ converges strongly to the variational inequality

$$\langle (I - f)x_0, x - x_0 \rangle \geq 0, \quad x \in A^{-1}(0). \quad (4.5)$$

5. CONCLUSION

We have shown that the convergence theorems for modified Krasnoselskii-Mann iteration with step size used for approximating the fixed points of total asymptotically nonexpansive mappings in Hilbert spaces could be derived from the corresponding convergence theorems in the class of nonexpansive-type mappings. Furthermore, we obtained a unifying technique of proof for various well known results in the fixed point theory of total asymptotically nonexpansive mappings in literature. The results clearly generalizes and extend some existing results in literature

STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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