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DOUBLE INERTIAL STEPS RELAXED PROJECTION ALGORITHM FOR PSEUDOMONOTONE VARIATIONAL INEQUALITIES

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Abstract. In this paper, we propose an outer approximation algorithm for solving pseudomonotone variational inequalities (VI) in Hilbert space. The algorithm incorporates a double inertial technique, resulting in the double inertial relaxed projection algorithm (DIRPA). DIRPA generalizes the algorithm proposed by Yao, Iyiola and Shehu [J Sci Comput, 2022, 90(71): 1-29] (YIS Alg for short). Unlike the YIS Alg, DIRPA allows both inertial step-sizes to be adaptively updated. By taking suitable parameters, the global weak convergence of DIRPA is established under the same assumptions with YIS Alg. Numerical experiments show the efficiency of DIRPA.

Keywords. Variational inequalities, Double inertial step, Subgradient extragradient algorithm, Pseudomonotone, Weak convergence.

1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space $\mathbb{H}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $F : \mathbb{H} \to \mathbb{H}$ be a cost operator. Consider the variational inequality VI(C, F), which is to find a point $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0, \forall y \in C. \tag{1.1}$$

The dual variation inequality of Problem (1.1) is to find a point $x^* \in C$ such that

$$\langle F(y), y - x^* \rangle \geq 0, \forall y \in C. \tag{1.2}$$

The solution set of Problem (1.1) and Problem (1.2) are denoted by S and S_D, respectively. If F is continuous on C, together with the fact that C is convex, by Minty lemma, we have $S_D \subseteq S$. Hence, if F is pseudomonotone and continuous on C, we have $S = S_D$.

VI have found applications in nonlinear analysis, economics and optimization; see, for example, [2, 1, 3]. So, many projection algorithms are proposed for solving VI under different monotonicity of F.

In 1964, Goldstein [4] proposed a simple and efficient projection algorithm:

$$x^{n+1} = P_C(x^n - \lambda F(x^n)).$$

The convergence of this algorithm requires F is strongly monotone and L-Lipschitz continuous on C. In order to relax the strongly monotonicity to pseudomonotonicity, the extragradient algorithm (EGA)

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was introduced by Korpelevich [5] with the following form:

$$\begin{cases} y^n = P_C(x^n - \lambda F(x^n)), \\ x^{n+1} = P_C(x^n - \lambda F(y^n)). \end{cases}$$

The global convergence of EGA requires F is pseudomonotone and Lipschitz continuous on C. Note that EGA needs two projections onto C in each iteration. This may affect the efficiency of EGA whenever the projection onto C is not so easy to implement.

To save one projection onto C per iteration, Solodov and Tseng [6], He [7] and Sun [8] proposed the projection and contraction algorithm (PCA); Tseng [9] proposed the forward-backward splitting algorithm. Later, Censor et al. [10] proposed the subgradient extragradient algorithm (SEGA) for pseudomonotone VI. The update scheme of SEGA is as follows:

$$\begin{cases} y^n = P_C(x^n - \lambda F(x^n)), \\ x^{n+1} = P_{T_n}(x^n - \lambda F(y^n)), \\ T_n := \{x \in \mathbb{H} : \langle x^n - \lambda F(x^n) - y^n, x - y^n \rangle \leq 0\}, \end{cases}$$

where $T_n \supseteq C$ is a half-space. Since the projection onto a half-space has an explicit formula, SEGA is also needing one projection onto C in each iteration.

By using the information of the previous iteration point and the current iterate point, Polyak [11] proposed inertia technique to accelerate the convergence speed of algorithm, where the inertial step-size is a constant number. Inertial technique was wildly applied in algorithms for solving variational inequalities; see, for example, [13, 14, 12]. To accelerate the convergence speed, some inertial algorithms with variable inertial step-size are proposed for VI; see, for example, [16, 19, 15, 17, 18].

Another effective modification for Lipschitz continuous VI involves using a self-adaptive step size, which avoids the computational cost of estimating the Lipschitz constant; see, for example, [21, 20]. Recently, Yao et al. [22] proposed an extragradient algorithm (YIS Alg for short) with double inertial steps for pseudomonotone VI:

$$\begin{cases} z^n = x^n + \delta(x^n - x^{n-1}), \\ \omega^n = x^n + \theta_n(x^n - x^{n-1}), \\ y^n = P_C(\omega^n - \lambda_n F(\omega^n)), \\ x^{n+1} = (1-\alpha_n)z^n + \alpha_n P_{T_n}(\omega^n - \lambda_n F(y^n)), \\ T_n := \{\omega \in \mathbb{H} : \langle \omega^n - \lambda_n F(\omega^n) - y^n, \omega - y^n \rangle \leq 0\}, \end{cases} \tag{1.3}$$

where λ_n is given by

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu||\omega^n - y^n||}{||F(\omega^n) - F(y^n)||}, \lambda_n\}, & \text{if } F(\omega^n) - F(y^n) \neq 0; \\ \lambda_n, & \text{otherwise}, \end{cases}$$

By taking suitable parameters θ_n, δ and α_n, YIS Alg is weak global convergence whenever F is L-Lipschitz continuous and pseudomonotone. Numerical results indicate superior efficiency over the single inertial algorithm.

Very recently, based on SEGA, Anh P N [23] proposed an outer approximate algorithm (see Algorithm 3.1 therein, here we call APN Alg for short) for solving pseudomonotone VI as follows:

$$\begin{cases} y^n = P_C(x^n - \lambda_n F(x^n)), \\ \omega^n = x^n - v\lambda_n F(y^n), \\ x^{n+1} = P_{T_n}(\omega^n), \\ T_n := \{\omega \in \mathbb{H} : \langle \omega^n - \lambda_n F(\omega^n) - y^n, \omega - y^n \rangle \leq \gamma_n \|\omega^n - y^n\|^2\}, \end{cases}$$

where $v \in (0, \min\{1, \frac{1}{L}\})$ with L is the Lipschitz constant of F, $\lambda_n \in (b, \min\{\frac{1}{L}, \sqrt{\frac{v}{L}}\})$ with $b > 0$, and $\gamma_n \in (0, \min\{\frac{1-\xi_n^2 L^2}{2}, \frac{1-vL-a}{2}\})$ with $a \in (0, 1 - vL)$. The sequence $\{x^n\}$ generated by APN Alg is globally strong convergence to a solution of VI whenever F is Lipschitz continuous and partially pseudomonotone, see Theorem 3.1 therein. However, APN Alg can not relaxed to SEGA (because it needs $\gamma_n > 0$).

Inspired by [10, 22, 23], we propose a double inertial steps relaxed projection algorithm (DIRPA) for pseudomonotone and Lipschitz continuous VI. Compare to APN Alg, DIRPA employs a self-adaptive step-size (see (3.2) below). This allows DIRPA without needing to know the Lipschitz constant of F. Additionally, double inertial technique is incorporated in DIRPA to accelerate convergence. Moreover, both inertial step-size are adaptively updated (see (3.1)) to enhance efficiency. DIRPA generalized YIS Alg, see Remark 3.1 below. By taking suitable parameters, the global weak convergence of DIRPA is established under the same assumptions with YIS Alg. Numerical experiments show the efficiency of DIRPA.

This paper is organized as follows: Section 2 recalls some definitions, preliminary results, and key lemmas that will be used in later convergence analysis. In Section 3, we present DIRPA and show its global weak convergence. Section 4 presents numerical experiments.

2. Preliminaries

Let $\mathbb{H}$ be a real Hilbert space, and let $\mathbb{R}^n$ be n-dimensional Euclidean space. Let $C \subseteq \mathbb{H}$ be a nonempty, closed and convex subset. The weak convergence of a sequence $\{x^n\}_{n=1}^{\infty}$ to x as $n \to \infty$ is denoted by $x^n \rightharpoonup x$ while the strong convergence of $\{x^n\}_{n=1}^{\infty}$ to x as $n \to \infty$ is denoted by $x^n \to x$.

Definition 2.1. A mapping F: $\mathbb{H} \to \mathbb{H}$ is said to be:

(1) $L-$Lipschitz continuous with $L > 0$ if
$$\|F(x) - F(y)\| \leq L\|x - y\|, \ \forall x, y \in \mathbb{H}.$$
(2) pseudomonootone on $\mathbb{H}$ if
$$\langle F(x), y - x\rangle \geq 0 \Rightarrow \langle F(y), y - x\rangle \geq 0, \ \forall x, y \in \mathbb{H}.$$
(3) monotone on $\mathbb{H}$ if
$$\langle F(x) - F(y), x - y\rangle \geq 0, \ \forall x, y \in \mathbb{H}.$$
(4) sequentially weakly continuous if for each sequence $\{x^n\}$ in $\mathbb{H}$, $\{x^n\}$ converges weakly to $x \in \mathbb{H}$ implies $\{F(x^n)\}$ converges weakly to $F(x)$.

Remark 2.2. It is routine to check that (iii)⇒(ii). However, the converse implications generally fail to hold.

Definition 2.3. Let $C \subseteq \mathbb{H}$ be a nonempty, closed and convex set, and $u \in \mathbb{H}$. We let $P_C(u)$ denote the metric projection of u onto C, that is
$$P_C(u) = argmin\{\|u - y\| : y \in C\}.$$
Hence, recall from [24] that for any $x \in \mathbb{H}$, we have
$$z = P_C(x) \iff z \in C \text{ and } \langle x - z, z - y\rangle \geq 0, \ \forall y \in C. \tag{2.1}$$
Furthermore, it follows that
$$\langle x - y, P_C(x) - P_C(y)\rangle \geq \|P_C(x) - P_C(y)\|^2, \ \forall x, y \in \mathbb{H}$$
and
$$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \ \forall x \in \mathbb{H}, \ y \in C. \tag{2.2}$$

Lemma 2.4. ([22]) *The following statements hold in* $\mathbb{H}$*:*

(1) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle,\ \forall x, y \in \mathbb{H}$;

(2) $\|\alpha x+\beta y\|^2 = \alpha(\alpha+\beta)\|x\|^2 + \beta(\alpha+\beta)\|y\|^2 - \alpha\beta\|x-y\|^2,\ \forall x, y \in \mathbb{H},\ \alpha, \beta \in \mathbb{R}$.

Lemma 2.5. ([25]) *Let* $\{\varphi_n\}, \{\delta_n\}$ *and* $\{\theta_n\}$ *be the sequences in* $[0, +\infty)$ *and there exists a real number* θ *with* $0 \leq \theta_n \leq \theta < 1, \forall n \in N$*, such that*

$$\varphi_{n+1} \leq \varphi_n + \theta_n(\varphi_n - \varphi_{n-1}) + \delta_n, \forall n \geq 1,\ \sum_{n=1}^{+\infty} \delta_n < +\infty.$$

Then the following assertions hold:

(1) $\sum_{n=1}^{+\infty} [\varphi_n - \varphi_{n-1}]_+ < \infty, where [t]_+ := \max\{t, 0\}$;

(2) *there exists* $\varphi_* \in [0, +\infty)$ *such that* $\lim_{n\to\infty} \varphi_n = \varphi^*$.

Lemma 2.6. ([26]) *Let* C *be a nonempty subset of* $\mathbb{H}$ *and let* $\{x^n\}$ *be a sequence in* $\mathbb{H}$ *such that the following two conditions hold:*

(1) *foy any* $x \in C$, $\lim_{n\to\infty} \|x^n - x\|$ *exists;*

(2) *every sequential weak cluster point of* $\{x^n\}$ *is in* C.

Then $\{x^n\}$ *converges weakly to a point in* C.

3. Algorithm and Its Convergence Analysis

In this section, we first introduce DIRPA for solving pseudomonotone VI. Then, we show the well-definedness and the global weak convergence of DIRPA under some mild assumptions.

Now, we introduce DIRPA as following algorithm.

Algorithm 3.1. ***Step 1.*** *Take* $\mu \in (0, 1)$, $\gamma < \frac{1-\mu}{2}$ *and* $\lambda_1 > 0$, $\{\delta_n\}$, $\{\theta_n\}$ *and* $\{\alpha_n\}$ *be satisfied the Assumptions 3.2(V)-(VI). Let* $x^0, x^1 \in \mathbb{H}$ *be given starting points. Set n:=1.*

Step 2. *Compute*

$$\begin{cases} z^n = x^n + \delta_n(x^n - x^{n-1}) \\ \omega^n = x^n + \theta_n\left(x^n - x^{n-1}\right) \\ y^n = P_C\left(\omega^n - \lambda_n F(\omega^n)\right) \end{cases} \tag{3.1}$$

where

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\mu\|\omega^n - y^n\|}{\|F(\omega^n)-F(y^n)\|}, \lambda_n\} & F(\omega^n) - F(y^n) \neq 0, \\ \lambda_n & \text{otherwise.} \end{cases} \tag{3.2}$$

If $\omega^n = y^n = x^n$*, then stop.*

Step 3. *Compute*

$$x^{n+1} = (1-\alpha_n)z^n + \alpha_n P_{T_n}(\omega^n - \lambda_n F(y^n)),\ n \geq 1.$$

where T_n *is the half-space defined by*

$$T_n := \{\omega \in \mathbb{H} : \langle \omega^n - \lambda_n F(\omega^n) - y^n, \omega - y^n\rangle \leq \gamma\|\omega^n - y^n\|^2\}$$

Step 4. *Set* $n := n+1$ *and go to Step 2.*

For the sake of convergence, we summarize the assumptions as follows:

Assumption 3.2. *(I) The* C *be a nonempty, closed, and convex subset of* $\mathbb{H}$.

(II) The mapping $F : \mathbb{H} \to \mathbb{H}$ *is pseudomonotone and Lipschitz continuous (without needing to know the Lipschitz constant) on* $\mathbb{H}$.

(III) The solution set S is nonempty.
(IV) The mapping $F : \mathbb{H} \to \mathbb{H}$ satisfies the following property: if $\{x^n\} \subset \mathbb{H}$ and $\{x^n\} \rightharpoonup z$, then $\|Fz\| \le \liminf_{n\to\infty} \|Fx^n\|$.
(V) $0 \le \theta_n \le \theta_{n+1} \le 1$.
(VI) $\varepsilon \in (2,\infty)$, $t \in (0,1)$, $0 < \alpha \le \alpha_n \le \alpha_{n+1} < \frac{1}{1+\varepsilon}$, $0 \le \delta_n \le \delta_{n+1} \le \min\{\frac{t\varepsilon}{(1+\varepsilon)(1-\alpha)}, \frac{t(\varepsilon-\sqrt{2\varepsilon})}{\varepsilon}, \theta_1\}$.

Remark 3.3. DIRPA reduces to YIS Alg by taking $\gamma = 0$, $\delta_n = \delta$, $\forall n$ and reduces to Algorithm 1 of Shehu, Dong and Jiang in [27] by taking $\gamma = \delta_n = 0$, $\forall n$.

Remark 3.4. If $x^n = \omega^n = y^n$, then we have $x^n = P_C(x^n - \lambda_n F(x^n))$ and further $x^n \in S$.

Remark 3.5. From the definition of T_n, y^n and the properties of the projection operator (see (2.1)), we have $C \subseteq T_n$.

Remark 3.6. Note from the definition of λ_{n+1} (see (3.2)) that $0 \le \lambda_{n+1} \le \lambda_n$, for all $n \ge 1$. Hence the limit $\lim_{n\to\infty} \lambda_n$ exists. Moreover, by using the similar analysis in [28] of Lemma 3.4, together with the Lipschitz continuous of F, we further obtain that

$$\lambda_n \ge \min\{\mu/L, \lambda_1\},\ \forall n \ge 1.$$

From Remark 3.4 - Remark 3.5, DIRPA is well-defined.

Next, we show some properties about the sequence generated by DIRPA.

Lemma 3.7. *Let $\{\delta_n\}$ be the sequence satisfied Assumption 3.2(VI). Then the following inequalities hold:*
(1) $\tau_n \ge \bar{\tau} > 0$ with $\tau_n := \varepsilon{\delta_n}^2 + \varepsilon - 2\varepsilon\delta_n - 2$ and $\bar{\tau} = \varepsilon(\frac{t(\varepsilon-\sqrt{2\varepsilon})}{\varepsilon})^2 + \varepsilon - 2\varepsilon\frac{t(\varepsilon-\sqrt{2\varepsilon})}{\varepsilon} - 2$;
(2) $\zeta_n < \bar{\zeta} < \frac{\varepsilon+1}{1+\varepsilon} = 1$, $\forall n$, where $\zeta_n := \delta_n(1-\alpha) + \frac{1}{1+\varepsilon}$ and $\bar{\zeta} := \frac{t\varepsilon+1}{1+\varepsilon}$, $\forall n$.

Proof. 1. Let $\tau_n := \varepsilon{\delta_n}^2 + \varepsilon - 2\varepsilon\delta_n - 2$. Since $\{\tau_n\}$ is decreasing on $\delta_n \in (0, \frac{\varepsilon-\sqrt{2\varepsilon}}{\varepsilon}]$, together with the fact that $\delta_n < \frac{t(\varepsilon-\sqrt{2\varepsilon})}{\varepsilon}$, we obtain that

$$\tau_n \ge \bar{\tau} = \varepsilon(\frac{t(\varepsilon-\sqrt{2\varepsilon})}{\varepsilon})^2 + \varepsilon - 2\varepsilon\frac{t(\varepsilon-\sqrt{2\varepsilon})}{\varepsilon} - 2 > \varepsilon(\frac{\varepsilon-\sqrt{2\varepsilon}}{\varepsilon})^2 + \varepsilon - 2\varepsilon\frac{\varepsilon-\sqrt{2\varepsilon}}{\varepsilon} - 2 = 0.$$

2. Let $\zeta_n := \delta_n(1-\alpha) + \frac{1}{1+\varepsilon}$. From Assumption 3.2(VI), we have $\delta_n < \frac{t\varepsilon}{(1+\varepsilon)(1-\alpha)}$. This implies that $\zeta_n < \frac{t\varepsilon+1}{1+\varepsilon} = \bar{\zeta} \le \frac{\varepsilon+1}{1+\varepsilon} = 1$. □

Lemma 3.8. *Suppose that Assumptions 3.2(I)-(IV) hold. Let $\{\omega^n\}$ be the sequence generated by Algorithm 3.1. If there exists a subsequence $\{\omega^{n_k}\}$ convergent weakly to $z \in \mathbb{H}$ and*

$$\lim_{k\to\infty} \|\omega^{n_k} - y^{n_k}\| = 0,$$

then $z \in S$.

Proof. Since $\omega^{n_k} \rightharpoonup z$, $k \to \infty$, $\lim_{k\to\infty} \|\omega^{n_k} - y^{n_k}\| = 0$, $\{y^n\} \subseteq C$ and C is a closed and convex set, we obtain that $z \in C$. So, without loss of generality, we suppose that $F(z) \neq 0$.
Recall from the definition of y^n and (2.1), we have

$$\langle \omega^{n_k} - \lambda_{n_k} F(\omega^{n_k}) - y^{n_k}, \omega - y^{n_k}\rangle \le 0,\ \forall \omega \in C.$$

This equivalent to

$$\frac{1}{\lambda_{n_k}}\langle \omega^{n_k} - y^{n_k}, \omega - y^{n_k}\rangle \le \langle F(\omega^{n_k}), \omega - y^{n_k}\rangle,\ \forall \omega \in C.$$

Consequently, we get

$$\frac{1}{\lambda_{n_k}}\langle \omega^{n_k} - y^{n_k}, \omega - y^{n_k}\rangle + \langle F(\omega^{n_k}), y^{n_k} - \omega^{n_k}\rangle \le \langle F(\omega^{n_k}), \omega - \omega^{n_k}\rangle,\ \forall \omega \in C. \tag{3.3}$$

Since $\{\omega^{n_k}\}$ is weakly convergent, $\{\omega^{n_k}\}$ is bounded. Then, by the Lipschitz continuity of F, $\{F(\omega^{n_k})\}$ is bounded. Since $\|\omega^{n_k} - y^{n_k}\| \to 0$, $\{y^{n_k}\}$ is also bounded, according to the definition of λ_{n+1} from Algorithm 3.1 and Remark 3.6, we have

$$\lambda_{n_k} \geq min\{\lambda_1, \frac{\mu}{L}\}.$$

Hence, we have $\frac{1}{\lambda_{n_k}}\langle \omega^{n_k} - y^{n_k}, \omega - y^{n_k}\rangle + \langle F(\omega^{n_k}), y^{n_k} - \omega^{n_k}\rangle \to 0, k \to \infty$. This together with (3.3) obtain that

$$\liminf_{k\to\infty}\langle F(\omega^{n_k}), \omega - \omega^{n_k}\rangle \geq 0,\ \forall \omega \in C. \tag{3.4}$$

Moreover, for any $\omega \in C$, we have

$$\langle F(y^{n_k}), \omega - y^{n_k}\rangle = \langle F(y^{n_k}) - F(\omega^{n_k}), \omega - \omega^{n_k}\rangle + \langle F(\omega^{n_k}), \omega - \omega^{n_k}\rangle + \langle F(y^{n_k}), \omega^{n_k} - y^{n_k}\rangle. \tag{3.5}$$

Since $\lim_{k\to\infty}\|\omega^{n_k} - y^{n_k}\| = 0$ and F is L-Lipschitz continuous on $\mathbb{H}$, we have

$$\lim_{k\to\infty}\|F(\omega^{n_k}) - F(y^{n_k})\| = 0. \tag{3.6}$$

Combining (3.4), (3.5) with (3.6), we get

$$\liminf_{k\to\infty}\langle F(y^{n_k}), \omega - y^{n_k}\rangle \geq 0,\ \forall \omega \in C.$$

Next, we show that $z \in S$, we choose a sequence $\{\varepsilon_k\}$ of positive numbers such that $\{\varepsilon_k\}$ is decreasing and convergent to 0. For each $k \geq 1$, there exists an integer $n_{N_k} > 0$ such that

$$\langle F(y^{n_j}), \omega - y^{n_j}\rangle + \varepsilon_k \geq 0.\ \forall j \geq n_{N_k},\ \forall \omega \in C. \tag{3.7}$$

Since ε_k is decreasing, it is easy to see that the sequence $\{n_{N_k}\}$ is increasing. Furthermore, for each $k \geq 1$, since $\{y^{n_{N_k}}\} \subseteq C$, we have $F(y^{n_{N_k}}) \neq 0$. Setting

$$\nu^{n_{N_k}} = \frac{F(y^{n_{N_k}})}{\|F(y^{n_{N_k}})\|^2},$$

we get $\langle F(y^{n_{N_k}}), \nu^{n_{N_k}}\rangle = 1$ for each $k \geq 1$. By (3.7), we conclude that, for each $k \geq 1$, we have

$$\langle F(y^{n_{N_k}}), \omega - y^{n_{N_k}}\rangle + \varepsilon_k\langle F(y^{n_{N_k}}), \nu^{n_{N_k}}\rangle = \langle F(y^{n_{N_k}}), \omega + \varepsilon_k\nu^{n_{N_k}} - y^{n_{N_k}}\rangle \geq 0,\ \forall \omega \in C.$$

Since F is pseudomonotone on $\mathbb{H}$, we have

$$\langle F(\omega + \varepsilon_k\nu^{n_{N_k}}), \omega + \varepsilon_k\nu^{n_{N_k}} - y^{n_{N_k}}\rangle \geq 0,\ \forall \omega \in C. \tag{3.8}$$

On the other hand, for all $\omega \in C$, we have

$$\begin{aligned}
&\langle F(\omega + \varepsilon_k\nu^{n_{N_k}}), \omega + \varepsilon_k\nu^{n_{N_k}} - y^{n_{N_k}}\rangle \\
=&\langle F(\omega + \varepsilon_k\nu^{n_{N_k}}) + F(\omega) - F(\omega), \omega + \varepsilon_k\nu^{n_{N_k}} - y^{n_{N_k}}\rangle \\
=&\langle F(\omega), \omega + \varepsilon_k\nu^{n_{N_k}} - y^{n_{N_k}}\rangle + \langle F(\omega + \varepsilon_k\nu^{n_{N_k}}) - F(\omega), \omega + \varepsilon_k\nu^{n_{N_k}} - y^{n_{N_k}}\rangle \\
=&\langle F(\omega), \omega - y^{n_{N_k}}\rangle + \varepsilon_k\langle F(\omega), \nu^{n_{N_k}}\rangle + \langle F(\omega + \varepsilon_k\nu^{n_{N_k}}) - F(\omega), \omega + \varepsilon_k\nu^{n_{N_k}} - y^{n_{N_k}}\rangle.
\end{aligned}$$

This together with (3.8) gets

$$\langle F(\omega), \omega - y^{n_{N_k}}\rangle \geq \langle F(\omega) - F(\omega + \varepsilon_k\nu^{n_{N_k}}), \omega + \varepsilon_k\nu^{n_{N_k}} - y^{n_{N_k}}\rangle - \langle F(\omega), \varepsilon_k\nu^{n_{N_k}}\rangle. \tag{3.9}$$

Now, we show that $\lim_{k\to\infty}\varepsilon\nu^{n_{N_k}} = 0$. From Assumption 3.2(IV) and the fact that $y^{n_k} \rightharpoonup z, k \to \infty$, we have

$$0 < \|F(z)\| \leq \liminf_{k\to\infty}\|F(y^{n_k})\|.$$

This together with $\{y^{n_{N_k}}\} \subseteq \{y^{n_k}\}$ and $\varepsilon_k \to 0, k \to \infty$ gives

$$0 \le \limsup_{k\to\infty} \|\varepsilon_k \nu^{n_{N_k}}\| = \limsup_{k\to\infty} \frac{\varepsilon_k}{\|F(y^{n_{N_k}})\|} \le \frac{\limsup_{k\to\infty} \varepsilon_k}{\liminf_{k\to\infty} \|F(y^{n_{N_k}})\|} = 0,$$

which implies that $\lim_{k\to\infty} \varepsilon_k \nu^{n_{N_k}} = 0$. Hence, by passing the limit $k \to \infty$ in (3.9), together with the fact that $y^{n_k} \rightharpoonup z, k \to \infty$ and F is Lipschitz continuous, we obtain that

$$\langle F(\omega), \omega - z\rangle \ge 0, \ \forall \omega \in C.$$

Therefore, we have $\langle F(z), \omega - z\rangle \ge 0$. This completes the proof. □

For the rest of this paper, we define

$$u^n := P_{T_n}(\omega^n - \lambda_n F(y^n)), \ \forall n \ge 1.$$

Next we prove the following lemma for $\{x_n\}$ generated by Algorithm 3.1.

Lemma 3.9. *Suppose that Assumption 3.2(I)-(VI) hold. Let $\{x^n\}$ be the sequence generated by Algorithm 3.1. Then the following statements hold.*

(1) For any $x^* \in S$ and any n, it holds that

$$\|u^n - x^*\|^2 \le \|\omega^n - x^*\|^2 - (1 - \frac{\mu\lambda_n}{\lambda_{n+1}})\|u^n - y^n\|^2 - (1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - 2\gamma)\|\omega^n - y^n\|^2. \tag{3.10}$$

(2) The sequences $\{x^n\}$, $\{\omega^n\}$, $\{z^n\}$ and $\{u^n\}$ are all bounded. Moreover, it holds that

$$\sum_{n=1}^{\infty} \|x^{n+1} - x^n\|^2 < +\infty,$$

and

$$\lim_{n\to\infty} \|x^{n+1} - x^n\| = \lim_{n\to\infty} \|x^n - \omega^n\| = \lim_{n\to\infty} \|\omega^n - y^n\| = 0. \tag{3.11}$$

(3) For any $x^* \in S$, $\lim_{n\to\infty} \|x^n - x^*\|$ exists.

Proof. 1. Let $x^* \in S$. Then $\langle F(x^*), y_n - x^*\rangle \ge 0$. Since F is pseudomonotone, we have

$$\langle F(y^n), y^n - x^*\rangle \ge 0.$$

Hence, we get

$$\langle F(y^n), x^* - u^n\rangle \le \langle F(y^n), y^n - u^n\rangle. \tag{3.12}$$

From the definition of T_n we get $\langle \omega^n - \lambda_n F(\omega^n) - y^n, u^n - y^n\rangle \le \gamma \|\omega^n - y^n\|^2$.
Therefore

$$\begin{aligned}\langle \omega^n - \lambda_n F(y^n) - y^n, u^n - y^n\rangle &= \langle \omega^n - \lambda_n F(\omega^n) - y^n, u^n - y^n\rangle + \lambda_n\langle F(\omega^n) - F(y^n), u^n - y^n\rangle \\ &\le \lambda_n\langle F(\omega^n) - F(y^n), u^n - y^n\rangle + \gamma \|\omega^n - y^n\|^2.\end{aligned} \tag{3.13}$$

Using (2.2) and the definition of T_n, such that $x^* \in S \subseteq C \subseteq T_n$, one gets

$$\begin{aligned}\|u^n - x^*\|^2 &\le \|\omega^n - \lambda_n F(y^n) - x^*\|^2 - \|\omega^n - \lambda_n F(y^n) - u^n\|^2 \\ &= \|\omega^n - x^*\|^2 - \|\omega^n - u^n\|^2 + 2\lambda_n\langle F(y^n), x^* - u^n\rangle.\end{aligned}$$

These together with (3.12), we have

$$\begin{aligned}
&\|u^n - x^*\|^2 \\
\leq &\|\omega^n - x^*\|^2 - \|\omega^n - u^n\|^2 + 2\lambda_n\langle F(y^n), y^n - u^n\rangle \\
= &\|\omega^n - x^*\|^2 - \|\omega^n - y^n + y^n - u^n\|^2 + 2\langle \lambda_n F(y^n), y^n - u^n\rangle \\
= &\|\omega^n - x^*\|^2 - \|\omega^n - y^n\|^2 - \|y^n - u^n\|^2 - 2\langle \omega^n - y^n, y^n - u^n\rangle + 2\langle \lambda_n F(y^n), y^n - u^n\rangle \\
= &\|\omega^n - x^*\|^2 - \|\omega^n - y^n\|^2 - \|y^n - u^n\|^2 + 2\langle \lambda_n F(y^n) + y^n - \omega^n, y^n - u^n\rangle \\
= &\|\omega^n - x^*\|^2 - \|\omega^n - y^n\|^2 - \|y^n - u^n\|^2 + 2\langle \omega^n - \lambda_n F(y^n) - y^n, u^n - y^n\rangle.
\end{aligned} \tag{3.14}$$

Using (3.13) and the Cauchy-Schiwarz inequality, we get that,

$$\begin{aligned}
2\langle \omega^n - \lambda_n F(y^n) - y^n, u^n - y^n\rangle &\leq 2\lambda_n\|F(\omega^n) - F(y^n)\|\|u^n - y^n\| + 2\gamma\|\omega^n - y^n\|^2 \\
&\leq \frac{2\mu\lambda_n}{\lambda_{n+1}}\|\omega^n - y^n\|\|u^n - y^n\| + 2\gamma\|\omega^n - y^n\|^2 \\
&\leq \frac{\mu\lambda_n}{\lambda_{n+1}}(\|\omega^n - y^n\|^2 + \|u^n - y^n\|^2) + 2\gamma\|\omega^n - y^n\|^2.
\end{aligned}$$

Combing this with (3.14), we deduce that

$$\begin{aligned}
&\|u^n - x^*\|^2 \\
\leq &\|\omega^n - x^*\|^2 - \|\omega^n - y^n\|^2 - \|y^n - u^n\|^2 + \frac{\mu\lambda_n}{\lambda_{n+1}}\left(\|\omega^n - y^n\|^2 + \|u^n - y^n\|^2\right) + 2\gamma\|w^n - y^n\|^2 \\
= &\|\omega^n - x^*\|^2 - (1 - \frac{\mu\lambda_n}{\lambda_{n+1}})\|u^n - y^n\|^2 - (1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - 2\gamma)\|\omega^n - y^n\|^2.
\end{aligned}$$

This completes the proof of (3.10).

2. Recall from Remark 3.6 that limit $\lim_{n\to\infty}\lambda_n$, we have $\lim\limits_{n\to\infty}\left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > 0$ and $\lim\limits_{n\to\infty}\left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - 2\gamma\right) = 1 - \mu - 2\gamma > 0$. This together with (3.10) implies that there exists a natural number $N \geq 1$ such that

$$\|u^n - x^*\| \leq \|\omega^n - x^*\|, \ \forall n \geq N. \tag{3.15}$$

Now, from the definition of $\{x_{n+1}\}$ and (3.15), for all $n \geq N$, we deduce that

$$\begin{aligned}
\|x^{n+1} - x^*\|^2 &= \|(1-\alpha_n)z^n + \alpha_n u^n - x^*\|^2 = \|(1-\alpha_n)(z^n - x^*) + \alpha_n(u^n - x^*)\|^2 \\
&= (1-\alpha_n)\|z^n - x^*\|^2 + \alpha_n\|u^n - x^*\|^2 - \alpha_n(1-\alpha_n)\|z^n - u^n\|^2 \\
&\leq (1-\alpha_n)\|z^n - x^*\|^2 + \alpha_n\|\omega^n - x^*\|^2 - \alpha_n(1-\alpha_n)\|z^n - u^n\|^2.
\end{aligned} \tag{3.16}$$

Furthermore, we have

$$\|u^n - z^n\| = \frac{1}{\alpha_n}\|x^{n+1} - z^n\|. \tag{3.17}$$

Combining (3.17) with (3.16), we now obtain

$$\begin{aligned}
\|x^{n+1} - x^*\|^2 \leq &(1-\alpha_n)\|z^n - x^*\|^2 + \alpha_n\|\omega^n - x^*\|^2 \\
&- \frac{1-\alpha_n}{\alpha_n}\|x^{n+1} - z^n\|^2, \ \forall n \geq N.
\end{aligned} \tag{3.18}$$

From (1.3) and Lemma 2.4(2), we get

$$\begin{aligned}&\|z^n - x^*\|^2\\ =&\|x^n + \delta_n(x^n - x^{n-1}) - x^*\|^2 = \|(1+\delta_n)(x^n - x^*) - \delta_n(x^{n-1} - x^*)\|^2\\ =&(1+\delta_n)\|x^n - x^*\|^2 - \delta_n\|x^{n-1} - x^*\|^2 + \delta_n(1+\delta_n)\|x^n - x^{n-1}\|^2.\end{aligned} \tag{3.19}$$

Similarly, we get,

$$\begin{aligned}&\|\omega^n - x^*\|^2\\ =&\|x^n + \theta_n(x^n - x^{n-1}) - x^*\|^2 = \|(1+\theta_n)(x^n - x^*) - \theta_n(x^{n-1} - x^*)\|^2\\ =&(1+\theta_n)\|x^n - x^*\|^2 - \theta_n\|x^{n-1} - x^*\|^2 + \theta_n(1+\theta_n)\|x^n - x^{n-1}\|^2,\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}\|x^{n+1} - z^n\|^2 &= \|x^{n+1} - (x^n + \delta_n(x^n - x^{n-1}))\|^2 = \|(x^{n+1} - x^n) - \delta(x^n - x^{n-1})\|^2\\ &= \|x^{n+1} - x^n\|^2 + {\delta_n}^2\|x^n - x^{n-1}\|^2 - 2\delta\langle x^{n+1} - x^{n-1}, x^n - x^{n-1}\rangle\\ &\geq \|x^{n+1} - x^n\|^2 + {\delta_n}^2\|x^n - x^{n-1}\|^2 - 2\delta_n\|x^{n+1} - x^{n-1}\|\|x^n - x^{n-1}\|\\ &\geq \|x^{n+1} - x^n\|^2 + {\delta_n}^2\|x^n - x^{n-1}\|^2 - \delta_n(\|x^{n+1} - x^n\|^2 + \|x^n - x^{n-1}\|^2)\\ &= (1-\delta_n)\|x^{n+1} - x^n\|^2 + ({\delta_n}^2 - \delta_n)\|x^n - x^{n-1}\|^2.\end{aligned} \tag{3.21}$$

By substituting (3.19), (3.20) and (3.21) into (3.18), we have

$$\begin{aligned}&\|x^{n+1} - x^*\|^2\\ \leq&(1-\alpha_n)[(1+\delta_n)\|x^n - x^*\|^2 - \delta_n\|x^{n-1} - x^*\|^2 + \delta_n(1+\delta_n)\|x^n - x^{n-1}\|^2]\\ &+ \alpha_n[(1+\theta_n)\|x^n - x^*\|^2 - \theta_n\|x^{n-1} - x^*\|^2 + \theta_n(1+\theta_n)\|x^n - x^{n-1}\|^2]\\ &- \frac{1-\alpha_n}{\alpha_n}[(1-\delta_n)\|x^{n+1} - x^n\|^2 + ({\delta_n}^2 - \delta_n)\|x^n - x^{n-1}\|^2]\\ =&(1-\alpha_n)(1+\delta_n)\|x^n - x^*\|^2 - \delta_n(1-\alpha_n)\|x^{n-1} - x^*\|^2 + \delta_n(1-\alpha_n)(1+\delta_n)\|x^n - x^{n-1}\|^2\\ &+ \alpha_n(1+\theta_n)\|x^n - x^*\|^2 - \alpha_n\theta_n\|x^{n-1} - x^*\|^2 + \alpha_n\theta_n(1+\theta_n)\|x^n - x^{n-1}\|^2\\ &- \frac{(1-\alpha_n)(1-\delta_n)}{\alpha_n}\|x^{n+1} - x^n\|^2 - \frac{(1-\alpha_n)({\delta_n}^2 - \delta_n)}{\alpha_n}\|x^n - x^{n-1}\|^2\\ =&(1 + \alpha_n\theta_n + \delta_n(1-\alpha_n))\|x^n - x^*\|^2 - (\delta_n(1-\alpha_n) + \alpha_n\theta_n)\|x^{n-1} - x^*\|^2\\ &+ (\delta_n(1-\alpha_n)(1+\delta_n) + \alpha_n\theta_n(1+\theta_n) - \frac{(1-\alpha_n)({\delta_n}^2 - \delta_n)}{\alpha_n})\|x^n - x^{n-1}\|^2\\ &- \frac{(1-\alpha_n)(1-\delta_n)}{\alpha_n}\|x^{n+1} - x^n\|^2.\end{aligned}$$

Let

$$\begin{aligned}B_n &:= \delta_n(1-\alpha_n)(1+\delta_n) + \alpha_n\theta_n(1+\theta_n) - \frac{(1-\alpha_n)({\delta_n}^2 - \delta_n)}{\alpha_n},\\ C_n &:= \frac{(1-\alpha_n)(1-\delta_n)}{\alpha_n}.\end{aligned}$$

We have

$$\begin{aligned}\|x^{n+1} - x^*\|^2 \leq& (1 + \alpha_n\theta_n + \delta_n(1-\alpha_n))\|x^n - x^*\|^2 - (\delta_n(1-\alpha_n) + \alpha_n\theta_n)\|x^{n-1} - x^*\|^2\\ &+ B_n\|x^n - x^{n-1}\|^2 - C_n\|x^{n+1} - x^n\|^2.\end{aligned} \tag{3.22}$$

Let

$$\Lambda^n := \|x^n - x^*\|^2 - (\delta_n(1-\alpha_n) + \alpha_n\theta_n)\|x^{n-1} - x^*\|^2 + B_n\|x^n - x^{n-1}\|^2.$$

From the definition Λ^n and using (3.22), we have

$$\begin{aligned}
&\Lambda^{n+1} - \Lambda^n \\
=&\|x^{n+1} - x^*\|^2 - (\delta_n(1-\alpha_{n+1}) + \alpha_{n+1}\theta_{n+1})\|x^n - x^*\|^2 + B_{n+1}\|x^{n+1} - x^n\|^2 - \|x^n - x^*\|^2 \\
&+ (\delta_n(1-\alpha_n) + \alpha_n\theta_n)\|x^{n-1} - x^*\|^2 - B_n\|x^n - x^{n-1}\|^2 \\
\leq&(1 + \alpha_n\theta_n + \delta_n(1-\alpha_n))\|x^n - x^*\|^2 - (\delta_n(1-\alpha_n) + \alpha_n\theta_n)\|x^{n-1} - x^*\|^2 + B_n\|x^n - x^{n-1}\|^2 \\
&- C_n\|x^{n+1} - x^n\|^2 - (\delta_n(1-\alpha_{n+1}) + \alpha_{n+1}\theta_{n+1})\|x^n - x^*\|^2 + B_{n+1}\|x^{n+1} - x^n\|^2 \\
&- \|x^n - x^*\|^2 + (\delta_n(1-\alpha_n) + \alpha_n\theta_n)\|x^{n-1} - x^*\|^2 - B_n\|x^n - x^{n-1}\|^2 \\
=&(\alpha_n\theta_n + \delta_n(1-\alpha_n) - \alpha_{n+1}\theta_{n+1} - \delta_n(1-\alpha_{n+1}))\|x^n - x^*\|^2 \\
&- C_n\|x^{n+1} - x^n\|^2 + B_{n+1}\|x^{n+1} - x^n\|^2 \\
=&(\alpha_n(\theta_n - \delta_n) - \alpha_{n+1}(\theta_{n+1} - \delta_n))\|x^n - x^*\|^2 - C_n\|x^{n+1} - x^n\|^2 + B_{n+1}\|x^{n+1} - x^n\|^2. \quad (3.23)
\end{aligned}$$

Now using the assumption $\alpha_n \leq \alpha_{n+1}$ and $0 \leq \delta_n \leq \theta_1 \leq \theta_n \leq \theta_{n+1}$, we deduce $0 \leq \theta_n - \delta_n \leq \theta_{n+1} - \delta_n$. Hence,

$$\alpha_n(\theta_n - \delta_n) - \alpha_{n+1}(\theta_{n+1} - \delta_n) \leq 0, \ \forall n \geq 1.$$

Using (3.23) we get

$$\begin{aligned}
\Lambda^{n+1} - \Lambda^n &\leq -C_n\|x^{n+1} - x^n\|^2 + B_{n+1}\|x^{n+1} - x^n\|^2 \\
&= -(C_n - B_{n+1})\|x^{n+1} - x^n\|^2. \quad (3.24)
\end{aligned}$$

By the definition of B_n, C_n, and (V), (VI) of Assumption 3.2, we get $\frac{(1-\alpha_n)}{\alpha_n} \geq \varepsilon$, $0 \leq \delta_n(1+\delta_n) \leq 2$ and $0 \leq \theta_{n+1}(1+\theta_{n+1}) \leq 2$, $\forall n \leq 1$. This together with the fact that the function $f(x) := \frac{1-x}{x}$ is decreasing on $\mathbb{R}^{++}$ gives $\frac{1-\alpha_{n+1}}{\alpha_{n+1}} \leq \frac{1-\alpha_n}{\alpha_n}$, then

$$\begin{aligned}
&C_n - B_{n+1} \\
=&\frac{(1-\alpha_n)(1-\delta_n)}{\alpha_n} - \delta_n(1-\alpha_{n+1})(1+\delta_n) - \alpha_{n+1}\theta_{n+1}(1+\theta_{n+1}) + \frac{(1-\alpha_{n+1})({\delta_n}^2 - \delta_n)}{\alpha_{n+1}} \\
\geq&\frac{(1-\alpha_n)(1-\delta_n)}{\alpha_n} - 2(1-\alpha_{n+1}) - 2\alpha_{n+1} + \frac{(1-\alpha_n)({\delta_n}^2 - \delta_n)}{\alpha_n} \\
\geq&\varepsilon(1-\delta_n) + \varepsilon({\delta_n}^2 - \delta_n) - 2 \\
=&\varepsilon{\delta_n}^2 + \varepsilon - 2\varepsilon\delta_n - 2. \quad (3.25)
\end{aligned}$$

From the Lemma 3.7, we conclude that $C_n - B_{n+1} > 0$. Then from (3.24) and (3.25), we have

$$\Lambda^{n+1} - \Lambda^n \leq -\tau_n\|x^{n+1} - x^n\|^2 \leq -\bar{\tau}\|x^{n+1} - x^n\|^2 \leq 0. \quad (3.26)$$

Therefore, the sequence $\{\Lambda^n\}$ is non-increasing. Furthermore,

$$\begin{aligned}
\Lambda^n &= \|x^n - x^*\|^2 - (\delta_n(1-\alpha_n) + \alpha_n\theta_n)\|x^{n-1} - x^*\|^2 + B_n\|x^n - x^{n-1}\|^2 \\
&\geq \|x^n - x^*\|^2 - (\delta_n(1-\alpha_n) + \alpha_n\theta_n)\|x^{n-1} - x^*\|^2. \quad (3.27)
\end{aligned}$$

It follows that

$$
\begin{aligned}
\|x^n - x^*\|^2 &\le (\delta_n(1-\alpha_n) + \alpha_n\theta_n)\|x^{n-1} - x^*\|^2 + \Lambda^n \\
&\le (\delta_n(1-\alpha_n) + \alpha_n)\|x^{n-1} - x^*\|^2 + \Lambda^n \\
&\le (\delta_n(1-\alpha) + \frac{1}{1+\varepsilon})\|x^{n-1} - x^*\|^2 + \Lambda^n \\
&\le \zeta_n\|x^{n-1} - x^*\|^2 + \Lambda^1 \\
&\le \bar{\zeta}\|x^{n-1} - x^*\|^2 + \Lambda^1 \qquad (3.28)\\
&\le \bar{\zeta}^2\|x^{n-2} - x^*\|^2 + (1+\bar{\zeta})\Lambda^1 \\
&\ \vdots \\
&\le \bar{\zeta}^n\|x^0 - x^*\|^2 + (1+\bar{\zeta}+\cdots+\bar{\zeta}^{n-1})\Lambda^1 \\
&\le \bar{\zeta}^n\|x^0 - x^*\|^2 + \frac{1}{1-\bar{\zeta}}\Lambda^1, \qquad (3.29)
\end{aligned}
$$

where the fifth inequality and the eighth inequality hold because $\zeta_n \le \bar{\zeta} < 1$ (see Lemma 3.7 and the definition of ζ_n). From (3.29), we have $\{\|x^n - x^*\|\}$ and further $\{x^n\}$ are all bounded.
Next, we show that

$$\lim_{n\to\infty}\|x^n - \omega^n\| = \lim_{n\to\infty}\|\omega^n - y^n\| = 0.$$

From (3.28), we have

$$-\bar{\zeta}\|x^{n-1} - x^*\|^2 \le -\bar{\zeta}\|x^{n-1} - x^*\|^2 + \|x^n - x^*\|^2 \le \Lambda^1.$$

Recall from (3.27), we have

$$
\begin{aligned}
-\Lambda^{n+1} &\le -\|x^{n+1} - x^*\|^2 + (\delta_{n+1}(1-\alpha_{n+1}) + \alpha_{n+1}\theta_{n+1})\|x^n - x^*\|^2 \\
&\le (\delta_{n+1}(1-\alpha_{n+1}) + \alpha_{n+1}\theta_{n+1})\|x^n - x^*\|^2 \\
&\le (\delta_{n+1}(1-\alpha) + \frac{1}{1+\varepsilon})\|x^n - x^*\|^2 \\
&= \zeta_{n+1}\|x^n - x^*\|^2 \\
&\le \bar{\zeta}\|x^n - x^*\|^2 \\
&\le \bar{\zeta}^{n+1}\|x^0 - x^*\|^2 + \frac{\bar{\zeta}}{1-\bar{\zeta}}\Lambda^1, \qquad (3.30)
\end{aligned}
$$

where the third inequality holds from Assumption 3.2(VI), the fourth inequality holds from Lemma 3.7 and the last inequality holds from (3.29).
Using (3.26) and (3.30), we obtain

$$
\begin{aligned}
\bar{\tau}\sum_{k=1}^{n}\|x^{k+1} - x^k\|^2 \le \Lambda^1 - \Lambda^{n+1} &\le \Lambda^1 + \bar{\zeta}^{n+1}\|x^0 - x^*\|^2 + \frac{\bar{\zeta}}{1-\bar{\zeta}}\Lambda^1 \\
&= \bar{\zeta}^{n+1}\|x^0 - x^*\|^2 + \frac{1}{1-\bar{\zeta}}\Lambda^1.
\end{aligned}
$$

Therefore, from Lemma 3.7, we get,

$$\sum_{n=1}^{\infty}\|x^{n+1} - x^n\|^2 \le \frac{1}{(1-\bar{\zeta})\bar{\tau}}\Lambda^1 < +\infty. \qquad (3.31)$$

Hence, we have

$$\lim_{n\to\infty} \|x^{n+1} - x^n\| = 0. \tag{3.32}$$

From (3.32), Assumption 3.2(V)(VI) and the definition of z^n and ω^n, we have

$$\begin{aligned}\|x^{n+1} - z^n\| &\le \|x^{n+1} - x^n\| + \|x^n - z^n\| = \|x^{n+1} - x^n\| + \delta_n \|x^n - x^{n-1}\| \\ &\le \|x^{n+1} - x^n\| + \theta_1 \|x^n - x^{n-1}\| \to 0,\ n \to \infty,\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}\|x^{n+1} - \omega^n\| &\le \|x^{n+1} - x^n\| + \|x^n - \omega^n\| \le \|x^{n+1} - x^n\| + \theta_n \|x^n - x^{n-1}\| \\ &\le \|x^{n+1} - x^n\| + \|x^n - x^{n-1}\| \to 0,\ n \to \infty.\end{aligned} \tag{3.34}$$

From (3.33) and (3.34), together with the boundness of $\{x^n\}$, we obtain $\{\omega^n\}$, $\{z^n\}$ are all bounded. Furthermore,

$$\begin{aligned}\|\omega^n - z^n\| &= \|x^n + \theta_n(x^n - x^{n-1}) - x^n - \delta_n(x^n - x^{n-1})\| \\ &\le \delta_n \|x^n - x^{n-1}\| + \theta_n \|x^n - x^{x-1}\| \\ &\le \theta_1 \|x^n - x^{n-1}\| + \|x^n - x^{x-1}\| \to 0,\ n \to \infty.\end{aligned} \tag{3.35}$$

By the definition of x^{n+1} and Assumption 3.2(VI), we get

$$\|x^{n+1} - z^n\| = \|(1 - \alpha_n) z^n + \alpha_n u^n - z^n\| = \alpha_n \|z^n - u^n\| \ge \alpha \|z^n - u^n\|. \tag{3.36}$$

Using (3.33) and (3.36), we have

$$\|z^n - u^n\| \le \frac{1}{\alpha} \|x^{n+1} - z^n\| \to 0,\ n \to \infty. \tag{3.37}$$

From (3.35) and (3.37), we get

$$\|\omega^n - u^n\| \le \|\omega^n - z^n\| + \|z^n - u^n\| \to 0,\ n \to \infty. \tag{3.38}$$

Moreover, together with the boundness of $\{\omega^n\}$, it follows that $\{u^n\}$ is bounded. By using (3.32) and (3.33), we have

$$\|x^n - z^n\| \le \|x^n - x^{n+1}\| + \|x^{n+1} - z^n\| \to 0,\ n \to \infty.$$

Hence, from (3.32) and (3.34), we see that

$$\|x^n - \omega^n\| \le \|x^n - x^{n+1}\| + \|x^{n+1} - \omega^n\| \to 0,\ n \to \infty.$$

Recall from (3.10) and the boundness of $\{\omega^n\}$, $\{u^n\}$, there exist $M_1 > 0$ such that

$$\begin{aligned}(1 - \frac{\mu\lambda_n}{\lambda_{n+1}} + 2\gamma)\|\omega^n - y^n\|^2 &\le \|\omega^n - x^*\|^2 - \|u^n - x^*\|^2 \\ &= (\|\omega^n - x^*\| + \|u^n - x^*\|)(\|\omega^n - x^*\| - \|u^n - x^*\|) \\ &\le M_1(\|\omega^n - x^*\| - \|u^n - x^*\|) \\ &\le M_1 \|\omega^n - u^n\|.\end{aligned}$$

Using this, together with (3.38) and the fact that $\lim_{n\to\infty} \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}} + 2\gamma\right) = 1 - \mu + 2\gamma > 0$, we see that

$$\lim_{n\to\infty} \|\omega^n - y^n\| = 0.$$

This completes the proof of (3.11).

3. From the definition of B_n and Assumption 3.2, we have

$$\begin{aligned} B_n &\leq \delta_n(1-\alpha_n)(1+\delta_n) + \frac{2}{1+\varepsilon} + \frac{(1-\alpha)(\delta_n - {\delta_n}^2)}{\alpha} \\ &\leq \theta_1(1-\alpha)(1+\theta_1) + \frac{2}{1+\varepsilon} + \frac{(1-\alpha)(\theta_1(1-\delta_1))}{\alpha}. \end{aligned}$$

Hence, for some $M_2 > 0$, we have

$$B_n \leq M_2.$$

Using (3.22), we get

$$\begin{aligned} &\|x^{n+1} - x^*\|^2 \\ \leq&(1+\alpha_n\theta_n + \delta_n(1-\alpha_n))\|x^n - x^*\|^2 - (\delta_n(1-\alpha_n) + \alpha_n\theta_n)\|x^{n-1} - x^*\|^2 \\ &+ B_n\|x^n - x^{n-1}\|^2 - C_n\|x^{n+1} - x^n\|^2 \\ \leq&\|x^n - x^*\|^2 + (\alpha_n + \delta_n(1-\alpha_n))(\|x^n - x^*\|^2 - \|x^{n-1} - x^*\|^2) + B_n\|x^n - x^{n-1}\|^2 \\ \leq&\|x^n - x^*\|^2 + (\alpha_n + \delta_n(1-\alpha_n))(\|x^n - x^*\|^2 - \|x^{n-1} - x^*\|^2) + M_2\|x^n - x^{n-1}\|^2. \end{aligned} \tag{3.39}$$

On the other hand, from the facts that $\alpha_n \leq \frac{1}{1+\varepsilon}$, $\delta_n < \frac{t\varepsilon}{(1+\varepsilon)(1-\alpha)}$ and $t \in (0,1)$ (see Assumption 3.2), we have

$$\alpha_n + \delta_n(1-\alpha_n) \leq \frac{1}{1+\varepsilon} + \frac{t\varepsilon}{(1+\varepsilon)(1-\alpha)}(1-\alpha) < 1.$$

Hence, by using Lemma 2.5 (set $\theta = \frac{1}{1+\varepsilon} + \frac{t\varepsilon}{(1+\varepsilon)(1-\alpha)}(1-\alpha)$), (3.39), (3.31), we have

$$\lim_{n\to\infty} \|x^n - x^*\| \, exists.$$

This completes the proof. □

Theorem 3.10. *Suppose that Assumption 3.2(I)-(VI) hold. Let $\{x^n\}$ be the sequence generated by Algorithm 3.1. Then $\{x^n\}$ converges weakly to a point $x^* \in S$.*

Proof. Recall from Lemma 3.9 that $\{x^n\}$ is bounded. Hence, we suppose that $\{x^{n_k}\} \subseteq \{x^n\}$ such that $x^{n_k} \rightharpoonup z^* \in \mathbb{H}$, $k \to \infty$. This together with (3.11) gets $\omega^{n_k} \rightharpoonup z^* \in \mathbb{H}$. Therefore, by using Lemma 3.8 and (3.11), we get $z^* \in S$. Hence, by using Lemma 2.6 (let S be C therein) and the fact that limit $\lim\limits_{n\to\infty} \|x^n - x^*\|$ exists for any $x^* \in S$, we obtain that $\{x^n\}$ converges weakly to an element in S. This completes the proof. □

4. Numerical Experiments

In this section, we test Algorithm 3.1 (DIRPA for short), Algorithm 1 of Shehu, Dong and Jiang in [27] (SDJ Alg for short), Algorithm 1 of Yao, Iyiola and Shehu in [22] (YIS Alg for short), Algorithm 3.1 of Anh in [23] (Anh Alg for short) for solving pseudomonotone VI. All codes were written in MATLAB R2023b and performed on a PC Desktop Intel(R) Core(TM) i7-8565U CPU @1.80GHz 1.99 GHz, RAM 8.00GB.

Example 4.1. This example is taken from [29] and has been considered by many authors for numerical experiments; see, for example, Malitsky and Semenov [30], Thong et al [12]. In there, $F(x) := Mx + q$, where $M = NN^T + S + D$, $N \in \mathbb{R}^{m\times m}$ is a randomly generated matrix, $S \in \mathbb{R}^{m\times m}$ is a randomly skew-symmetric matrix, $D \in \mathbb{R}^{m\times m}$ is a randomly positive definite diagonal matrix (hence M is positive definite), q is a randomly vector in $\mathbb{R}^m$, and

$$C := \{x \in \mathbb{R}^m : Bx \leq b\},$$

where $B \in \mathbb{R}^{k\times m}$ is a random matrix, $b \in \mathbb{R}^k$ is a random vector with nonnegative entries.

So, $F(\cdot)$ is strongly monotone and Lipschitz continuous on $\mathbb{R}^m$ with Lipschitz constant $L = \|M\|$. The projection onto C is computed by Matlab solver "quadprog". Let $q = 0$, then $x^* = 0$ is the unique solution of this VI. Hence, the stopping criterion for DIRPA, SDJ Alg, YIS Alg and Anh Alg are all $\text{dist}(x^n, S) \leq 10^{-4}$, i.e.,

$$\|x^n\| \leq 10^{-4}.$$

The parameters for YIS Alg, SDJ Alg and Anh Alg in Table 1 follow the suggestions in their papers. We report the number of iterations (Iter), CPU times in seconds, the dimensional number m for different inequality constraints in Table 2 and Table 3, averaged 5 instance.

TABLE 1. Parameters for DIRPA, YIS Alg, SDJ Alg and Anh Alg

DIRPA	$\lambda_1 = 0.1$	μ=0.9	θ_n=1	α_n=0.2903	$\delta = 0.05$
	$t = 0.99$	γ=0.004	$\delta_n = \frac{n}{n+1}\delta$		
YIS Alg	$\lambda_1 = 0.1$	μ=0.9	θ_n=1	α_n=0.2903	δ=0.0241
SDJ Alg	$\lambda_1 = 0.1$	μ=0.9	θ_n=1	α_n=0.2903	
Anh Alg	$\tau = \min\{\frac{1}{L}, \sqrt{\frac{v}{L}}\}$	b=0.01τ	v=0.45α	α=min$\{1,\frac{1}{L}\}$	$\delta_n = b + \frac{b}{5n+10}$
	$\gamma_n = 0.02\delta_n$				

TABLE 2. Results for Example 5.1 with $k = 10, m = 20$

	Iter	CPU
DIRPA	761	1.4278
YIS Alg	820	1.9256
SDJ Alg	963	2.4261
Anh Alg	804203	592.995

TABLE 3. Results for Example 5.1 with $k = 50$

	m=60		m=80	
	Iter	CPU	Iter	CPU
DIRPA	2423	11.4545	5069	26.0915
YIS Alg	2734	12.9789	5533	32.5329
SDJ Alg	3768	13.1210	7395	45.1865

Example 4.2. To test high-dimensional VI, we modify the feasible set C in Example 4.1 as the nonnegative orthant, that is $C = \mathbb{R}^m_+$. Let the mapping F be defined by Example 4.1. We define the feasible set C by $C := \{x \in \mathbb{R}^m_+\}$. In this case, $x^* = 0$ is also the unique solution of this VI. So, the stopping criterion is also taken $\|x^n\| \leq 10^{-4}$. We report Iter, CPU, different m in Table 4, averaged 5 instances.

Remark 4.1. From Table 2, we see that all algorithms can find the solution of VI, Anh Alg requires much more CPU time. So, in the rest numerical experiments, we only test DIRPA, YIS Alg and SDJ Alg. From Table 2 - Table 4, we see that DIRPA has the fewer total number of iterations and CPU time.

TABLE 4. Results for Example 5.2 with different m

	m=1000		m=3000	
	Iter	CPU	Iter	CPU
DIRPA	2666	1.1409	3103	32.9787
YIS Alg	2719	1.1950	3167	34.0118
SDJ Alg	4699	1.9999	5020	51.2496

5. Conclusion

In this paper, we propose an outer approximation algorithm DIRPA for solving pseudomonotone VI in Hilbert space. DIRPA generalizes the Algorithm 3.1 of Yao, Iyiola and Shehu in [22]. Moreover, DIRPA allows both inertial step-sizes to be adaptively updated. By taking suitable parameters, the global weak convergence of DIRPA is established under the same assumptions in [22]. Numerical experiments show the efficiency of DIRPA.

Statements and Declarations

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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