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OUTPUT FEEDBACK ROBUST H_∞ CONTROL DESIGN OF ELLIPTICAL ORBITAL SPACECRAFT RENDEZVOUS SYSTEM WITH INPUT SATURATION

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Abstract. This paper studies the observer-based output feedback robust H_∞ control design problem for elliptical orbital spacecraft rendezvous system subject to both input saturation and external disturbance. Due to the limitation of existing technology and the effect of external disturbance, it is difficult to accurately measure the relative velocity information between target spacecraft and chaser spacecraft. In order to solve this problem, a state observer of spacecraft rendezvous system is established by the nonlinear Riccati matrix inequality method. Based on the obtained state observer, a discrete dynamic gain scheduling approach is adopted to design an observer-based output feedback robust H_∞ control of spacecraft rendezvous system, which can be obtained by solving the parametric periodic Riccati-like equation. The obtained observer-based output feedback robust H_∞ control can not only improve the dynamic performance of elliptical orbital rendezvous system, but also minimize the effect of external disturbance on the system. Finally, a practical example is provided to show the effectiveness of the proposed control design approach.

Keywords. Spacecraft rendezvous, State observer, Gain scheduling control approach, Robust H_∞ control, Input saturation.

1. Introduction

Spacecraft rendezvous has been regarded as an important technology in many aerospace engineer such as space laboratories, space stations, remote sensing platforms and space communications. In these space facilities, successful spacecraft rendezvous is the precondition for accomplishing many space flight missions such as space transporting, rescue, docking, repairing mal-functional spacecraft, refueling satellite on orbit and so on [18]. Therefore, the research of spacecraft rendezvous not only has important practical significance, but also has wide application value.

In the spacecraft rendezvous, consider two spacecrafts, where one is the target spacecraft, the other is the chaser spacecraft. When the target spacecraft is moving in a circular orbit, the relative motion of two spacecrafts can be described by nonlinear time-invariant system [17], whose linearized system at the origin is linear time-invariant system, referred to as C-W system [3]. When the target spacecraft is moving in an elliptical orbit, the relative motion of two spacecrafts can be described by nonlinear time-varying system [24], whose linearized system at the origin are linear time-varying system, referred to as T-H system [2]. From the existing research results, it can be seen that C-W system and T-H system are both the basic tool and foundation for the spacecraft rendezvous.

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In the past few decades, most spacecraft rendezvous control problems have been studied for C-W system. However, T-H system is rarely applied in the research of spacecraft rendezvous. This is mainly because T-H system is not only time-varying , but also its time-varying coefficients are all implicit functions, which makes spacecraft rendezvous much more complicated. This is the first motivation for this research.

Due to limitations in thrust equipment and engine power, real actuators cannot provide unlimited thrust[32, 25, 16]. Thus, input constraint is inevitable in the spacecraft rendezvous system. In recent few years, some input-constrained elliptical orbital spacecraft rendezvous problems have been studies by different control design methods. For example, the periodic Lyapunov equation method is proposed in [30, 7] to solve the input-constrained elliptical orbital rendezvous problem. In [29], the periodic Riccati equation approach is adopted to achieve the more complicated elliptical orbital spacecraft rendezvous subject to both the energy saturation and the control magnitude constraint. In [7, 29, 30], different full state feedback controls are designed, while the relative position and velocity information are both assumed to be available. In practice, it is difficult to measure accurately the velocity information of spacecraft rendezvous system due to the technical constraint of device and the disturbance of external environment. In order to solve this problem, the periodic Lyapunov equation method is used in [14, 28] respectively to design two different types of state observer to solve the global stabilization problem of the neutrally stable linear periodic system with bounded control constraints. Since the spacecraft rendezvous system is inevitably affected by the unknown external disturbances, such as J_2 perturbation, solar radial pressure force, aerodynamic force and magnetic torque[11]. Under these external disturbances, periodic Riccati equation approach is further utilized in [6] to study the robust H_∞ control problem of the elliptical orbital rendezvous system. From research results mentioned above, it can be seen there are no existing results to study the estimation problem of elliptical orbital spacecraft rendezvous with both input constraint and the external disturbance. This motivates us to search for a novel control approach to design the observer-based output feedback robust H_∞ control of elliptical orbital rendezvous system for both high control precision and good convergence performance. This is the second motivation for this research.

Gain scheduling control approach is one of the most important nonlinear control approaches, which includes static gain scheduling control approach, continuous dynamic gain scheduling control approach, and discrete dynamic gain scheduling control approach. These approaches have been widely used in control design problem of the input saturated spacecraft rendezvous system. For example, a static gain scheduling control approach is proposed in [21] to design state feedback control to solve the control design problem of the input saturated circular orbit spacecraft rendezvous system. In order to improve the dynamic performance of system, a continuous dynamic gain scheduling control approach is used in [26, 31] to design the state feedback control of the input saturated circular orbit spacecraft rendezvous system. Meanwhile, the discrete gain scheduling control approach is adopted in [20, 22] to design the state feedback control and robust state feedback control of the input saturated circular orbit spacecraft rendezvous system, respectively. Since the relative velocity information of system can not be accurately measured, the discrete gain scheduling control approach is utilized in [23] to solve the observer-based output feedback control problem for the saturated input circular orbit spacecraft rendezvous system. Under external disturbances, the discrete gain scheduling control approach is further applied in [10] to solve the observer-based output feedback robust H_∞ control problem for the input saturated circular orbit rendezvous system. In addition, the discrete gain scheduling control approach is also applied to respectively design the state feedback control of the input saturated elliptical orbit rendezvous system [5]. In the above literature [5, 10, 20, 22, 21, 23, 26, 31], different gain scheduling control approaches are mainly utilized to design state (output) feedback (robust H_∞) control to solve the input saturated circular orbit spacecraft rendezvous problem. Nevertheless, only there are few study results to solve state (output) feedback control problem of the input saturated elliptical orbit spacecraft rendezvous system [5]. To the best of our knowledge, there are no theoretical results to study the dynamic performance improvement problem of the input saturated elliptical orbit spacecraft rendezvous system subject to both state estimation and external disturbances. This prompts us to adopt the dynamic gain scheduling approach to study the observer-based output feedback robust H_∞ control problem of the input saturated elliptical orbital spacecraft rendezvous system.

In this paper, the discrete gain scheduling approach is adopted to design the observer-based output feedback robust H_∞ control for the dynamic performance improvement of the input saturated elliptical orbital rendezvous system. There are three innovations in this paper: (i) A novel Riccati matrix inequality is established to solve the observer design problem of the input saturated elliptical orbital spacecraft rendezvous system. (ii) The rotation (reflection) transformation technique is used to construct time-invariant elliptic invariant sets to determine the switching points of some discrete parameters. As these discrete parameters gradually increase, an observer-based output feedback robust H_∞ control including these discrete parameters is designed to improve the dynamic performance of the input saturated elliptical orbital rendezvous system. (iii) Compared with the continuous gain scheduling control approach, the discrete gain scheduling control approach can guarantee the periodic Riccati-like equation has a unique positive definite periodic solution, which can be solved by a new numerical algorithm. This obtained periodic solution is used to design the observer-based output feedback robust H_∞ control of the input saturated elliptical orbital rendezvous system.

The remainder of this paper is organized as follows. The input saturated elliptical orbital rendezvous system is transformed into a linear periodic system in Section 2, and the observer-based output feedback robust H_∞ control design problem for the linear periodic system is formulated. Section 3 presents some preliminary results. Based on these preliminary results, the discrete gain scheduling control approach is used to design the observer-based output feedback robust H_∞ control of the linear periodic system in Section 4. To show the effectiveness of the results, the simulation results are provided in Section 5. Finally, Section 6 concludes the paper.

2. Problem Formulation

The rendezvous process of the target spacecraft and the chaser spacecraft is depicted in Figure 1. Let

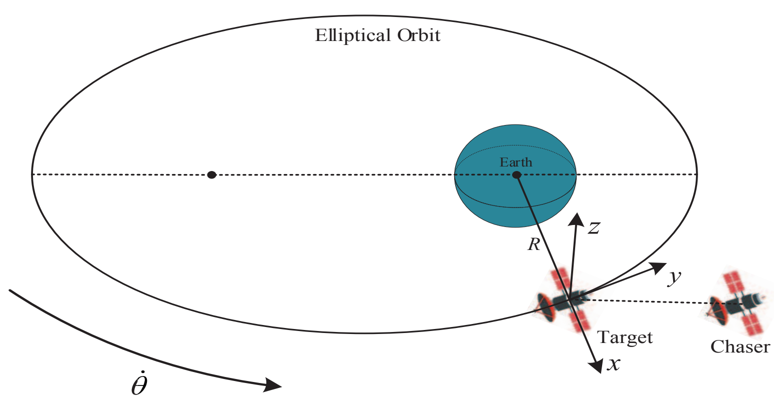

Figure 1. Spacecraft rendezvous system

h be the orbital angular momentum of the target, $e \in [0, 1)$ is the eccentric orbit, μ is the gravitational constant of the Earth, and θ is the true anomaly. Then $\dot{\theta} = k^2(1 + e\cos\theta)^2$, where $k = \mu/h^{3/2}$ is a constant. In the rotating coordinate system of Figure 1, if the distance between the chaser spacecraft and the target spacecraft is much smaller than the distance from the target spacecraft to the center of the gravity field, then the relative dynamic model can be described by the following input saturated linear time-varying system

$$\left[\begin{array}{c} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{array}\right] = \left[\begin{array}{c} 2k\dot{\theta}^{3/2}x(t) + 2\dot{\theta}\dot{y}(t) + \dot{\theta}^2 x(t) + \ddot{\theta}y(t) \\ -k\dot{\theta}^{3/2}y(t) - 2\dot{\theta}\dot{x}(t) + \dot{\theta}^2 y(t) - \ddot{\theta}x(t) \\ -k\dot{\theta}^{3/2}z(t) \end{array}\right] + \mathrm{sat}(u(t)) + \omega(t) \tag{2.1}$$

which is suffered from the effects of external disturbances. the above system (2.1) is also called the known T-H system with both input saturation and external disturbances, where $[x(t), y(t), z(t)]$ is the relative position of the chaser spacecraft, $[\dot{x}(t), \dot{y}(t), \dot{z}(t)]$ and $[\ddot{x}(t), \ddot{y}(t), \ddot{z}(t)]$ denote, respectively, the first and second order derivatives of the relative position $[x(t), y(t), z(t)]$ with respect to the time t, $u(t) = [u_1(t), u_2(t), u_3(t)]^\top$ is the acceleration vector induced by the thrust force on the chaser spacecraft, and $\omega(t) = [\omega_1(t), \omega_2(t), \omega_3(t)]^\top$ is a bounded external disturbance vector of the chaser spacecraft, i.e., $\|\omega\| \leq d_\omega$, d_ω is a given positive constant.

In order to reduce system (2.1), define

$$\xi(t) = [x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \dot{z}(t)]^{\mathrm{T}}$$

Then system (2.1) can be expressed as the following compact form

$$\dot{\xi}(t) = \mathbb{A}(t)\xi(t) + \mathbb{B}(t)\mathrm{sat}(u(t)) + \mathbb{B}_\omega(t)\omega(t). \tag{2.2}$$

where

$$\mathbb{A}(t) = \left[\begin{array}{cc} 0_3 & I_3 \\ \mathbb{A}_1(t) & \mathbb{A}_2(t) \end{array}\right], \quad \mathbb{B}(t) = \mathbb{B}_\omega(t) = \left[\begin{array}{c} 0_3 \\ I_3 \end{array}\right]$$

and

$$\mathbb{A}_1(t) = \left[\begin{array}{ccc} \dot{\theta}^2 + 2k\dot{\theta}^{3/2} & \ddot{\theta} & 0 \\ -\ddot{\theta} & \dot{\theta}^2 - k\dot{\theta}^{3/2} & 0 \\ 0 & 0 & -k\dot{\theta}^{3/2} \end{array}\right], \quad \mathbb{A}_2(t) = \left[\begin{array}{ccc} 0 & 2\dot{\theta} & 0 \\ -2\dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

In system (2.2), the true anomaly θ is the implicit expression of the time t, which will bring the difficult application of system (2.2). In order to solve this difficult problem, the true anomaly θ is selected as an independent variable instead of the time t. $v'(\theta)$ is denoted as the derivative of $v(\theta)$ with respect to θ. The following transformations are made:

$$[\bar{x}(\theta), \bar{y}(\theta), \bar{z}(\theta)] = \rho(\theta)[x(t), y(t), z(t)],$$
$$[\bar{u}_1(\theta), \bar{u}_2(\theta), \bar{u}_3(\theta)] = [u_1(t), u_2(t), u_3(t)]$$

and

$$[\bar{\omega}_1(\theta), \bar{\omega}_2(\theta), \bar{\omega}_3(\theta)] = [\omega_1(t), \omega_2(t), \omega_3(t)]$$

where $\rho(\theta) = 1 + e\cos\theta$.

Let $\boldsymbol{x}(\theta)$, $\boldsymbol{u}(\theta)$ and $\boldsymbol{\omega}(\theta)$ defined below be new state, control and disturbance vectors, respectively

$$\boldsymbol{x}(\theta) = [\bar{x}(\theta), \bar{y}(\theta), \bar{z}(\theta), \bar{x}'(\theta), \bar{y}'(\theta), \bar{z}'(\theta)],$$
$$\boldsymbol{u}(\theta) = [\bar{u}_1(\theta), \bar{u}_2(\theta), \bar{u}_3(\theta)]$$

and

$$\boldsymbol{\omega}(\theta) = [\bar{\omega}_1(\theta), \bar{\omega}_2(\theta), \bar{\omega}_3(\theta)]$$

By using the linear transformation

$$\boldsymbol{x}(\theta) = T(\theta)\boldsymbol{\xi}(t) \tag{2.3}$$

where

$$T(\theta) = \left[\begin{array}{cc} \rho(\theta)I_3 & 0 \\ -e\sin\theta I_3 & \dfrac{1}{k^2\rho(\theta)}I_3 \end{array}\right].$$

The system described by (2.2) is transformed into the following linear periodic system:

$$\boldsymbol{x}'(\theta) = A(\theta)\boldsymbol{x}(\theta) + B(\theta)\text{sat}(\boldsymbol{u}(\theta)) + B_\omega(\theta)\boldsymbol{\omega}(\theta) \tag{2.4}$$

where $A(\theta)$, $B(\theta)$ and $B_\omega(\theta)$ are given, respectively, by

$$A(\theta) = \begin{bmatrix} 0_3 & I_3 \\ A_1(\theta) & A_2(\theta) \end{bmatrix}, B(\theta) = B_\omega(\theta) = \frac{1}{k^4\rho^3(\theta)} \begin{bmatrix} 0_3 \\ I_3 \end{bmatrix} \tag{2.5a}$$

$$A_1(\theta) = \begin{bmatrix} \dfrac{3}{\rho(\theta)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2(\theta) = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.5b}$$

Since the relative position $[x(t), y(t), z(t)]$ of spacecraft rendezvous can be measured accurately, the measurement output of systems (2.4) can be defined as

$$\boldsymbol{y}(\theta) = C(\theta)\boldsymbol{x}(\theta), \quad C(\theta) = [I_3,\ 0] \tag{2.6}$$

This, together with (2.4), yields that the following spacecraft rendezvous system with the measurement output

$$\begin{cases} \boldsymbol{x}'(\theta) = & A(\theta)\boldsymbol{x}(\theta) + B(\theta)\text{sat}(\boldsymbol{u}(\theta)) + B_\omega(\theta)\boldsymbol{\omega}(\theta) \\ \boldsymbol{y}(\theta) = & C(\theta)\boldsymbol{x}(\theta) \end{cases} \tag{2.7}$$

For the spacecraft rendezvous system (2.7), the following regulated output vector is introduced

$$\boldsymbol{z}(\theta) = C_0(\theta)\boldsymbol{x}(\theta) + D_0(\theta)\boldsymbol{u} \tag{2.8}$$

where

$$C_0(\theta,\gamma) = \frac{\gamma}{\rho(\theta)} \begin{bmatrix} I_3 & 0 \end{bmatrix}, \quad D_0(\theta) = \begin{bmatrix} 0 \\ \sqrt{r(\theta)} I_3 \end{bmatrix}$$

are, respectively, 2π-periodic matrices, and $D_0^\top(\theta)C_0(\theta) = 0$, $\gamma > 0$, $r(\theta) > 0$. Based on this regulated output vector (2.8), robust H_∞ performance of the spacecraft rendezvous system (2.7) is defined as follows:

Definition 2.1. The spacecraft rendezvous system (2.7) has robust H_∞ performance for the external disturbance, if the following conditions are satisfied:

- **(i):** The spacecraft rendezvous system (2.7) is asymptotically stable for the external disturbances $\omega = 0$.
- **(ii):** With zeros initial condition and a prescribed disturbance attenuated level $\gamma_\omega > 0$, the following integral inequality holds

$$J = \int_{\theta_0}^{+\infty} \boldsymbol{x}^\top(\theta)Q_0(\theta,\gamma)\boldsymbol{x}(\theta) + \boldsymbol{u}^\top(\theta)R_0(\theta)\boldsymbol{u}(\theta) - \gamma_\omega^2\boldsymbol{\omega}^\top(\theta)\boldsymbol{\omega}(\theta)\mathrm{d}\theta \le 0 \tag{2.9}$$

where $Q_0(\theta,\gamma) = C_0^\top(\theta,\gamma)C_0(\theta,\gamma) \ge 0$, $R_0(\theta) = D_0^\top(\theta)D_0(\theta) > 0$.

In practical application, the relative velocity measurement of spacecraft rendezvous is accomplished indirectly. The relative position is first read by the sensor, then the relative velocity is obtained by numerical approximate computation. Such indirect measurement is not only costly and difficult, but also it can not be made accurately. As the result, the obtained relative velocity affects the safe and high-quality control of spacecraft rendezvous. In order to estimate the relative velocity of spacecraft rendezvous, the state observer and its related output feedback control are considered as follows:

$$\bar{\boldsymbol{x}}'(\theta) = A(\theta)\bar{\boldsymbol{x}}(\theta) + B(\theta)\text{sat}(\boldsymbol{u}(\theta)) - L(\theta)[\boldsymbol{y}(\theta) - \bar{\boldsymbol{y}}(\theta)] \tag{2.10}$$

and

$$\boldsymbol{u}(\theta) = K(\theta)\bar{\boldsymbol{x}}(\theta), \tag{2.11}$$

where $\bar{\boldsymbol{x}}(\theta)$ is the estimate vector of the state $\boldsymbol{x}(\theta)$, $L(\theta)$ and $K(\theta)$ are respectively the observer gain matrix and the output control gain matrix, which are to be determined later.

Let $\boldsymbol{e}(\theta) = \boldsymbol{x}(\theta) - \bar{\boldsymbol{x}}(\theta)$ be the estimation error of the observer. After simple calculation, we have

$$\dot{\boldsymbol{e}}(\theta) = [A(\theta) + L(\theta)C(\theta)]\boldsymbol{e}(\theta) + B_\omega(\theta)\boldsymbol{\omega}(\theta) \tag{2.12}$$

When $\boldsymbol{\omega}(\theta) = 0$, system (2.12) can be simplified as

$$\boldsymbol{e}'(\theta) = [A(\theta) + L(\theta)C(\theta)]\boldsymbol{e}(\theta) \tag{2.13}$$

For system (2.13), our goal is to design the observer gain matrix $L(\theta)$ such that system (2.13) is stable. it is sufficient to show that there exists a positive definite matrix $P_e(\theta)$ such that the following linear time varying matrix inequity is satisfied.

$$P_e(\theta)[A(\theta) + L(\theta)C(\theta)] + [A(\theta) + L(\theta)C(\theta)]^\top P_e(\theta) \leq -\gamma_e I_6, \quad \gamma_e > 0 \tag{2.14}$$

In the matrix inequity (2.14), the matrices $A(\theta)$ and $C(\theta)$ have the special form: (2.5a) and (2.5b). It is easy to find $L(\theta)$ such that $A(\theta)+L(\theta)C(\theta)$ is a constant Hurwitz matrix. Thus, there exists a constant positive definite matrix $P_e(\theta)$, which is independent on θ, denoted as P_e, such that (2.14) holds. This overcomes the difficulty of the obtained observer gain matrix $L(\theta)$ by solving the linear time-varying inequality (2.14).

From (2.13), it can be seen that the estimation error of the observer can be determined by the observer gain $L(\theta)$, which is independent of the output control gain matrix $K(\theta)$. This implies that the separation principle in [19] can be applied to the observer-based output feedback robust H_∞ control design of the spacecraft rendezvous system (2.7) with input saturation. Define $\boldsymbol{\zeta}(\theta) = [\boldsymbol{x}^\top(\theta), \boldsymbol{e}^\top(\theta)]^\top$, it follows from (2.4) and (2.12) that the augmented system is

$$\begin{cases} \dot{\boldsymbol{\zeta}}(\theta) = \mathcal{A}(\theta)\boldsymbol{\zeta}(\theta) + \mathcal{B}(\theta)\mathrm{sat}(\boldsymbol{u}(\theta)) + \mathcal{B}_\omega(\theta)\boldsymbol{\omega}(\theta) \\ \boldsymbol{y}(\theta) = \mathcal{C}(\theta)\boldsymbol{\zeta}(\theta) \end{cases} \tag{2.15}$$

where

$$\mathcal{A}(\theta) = \begin{bmatrix} A(\theta) & 0 \\ 0 & A(\theta) + L(\theta)C(\theta) \end{bmatrix}, \mathcal{B}(\theta) = \begin{bmatrix} B(\theta) \\ 0 \end{bmatrix}$$
$$\mathcal{B}_\omega(\theta) = \begin{bmatrix} B_\omega(\theta) \\ B_\omega(\theta) \end{bmatrix}, \mathcal{C}(\theta) = \begin{bmatrix} C(\theta) & 0 \end{bmatrix}$$

From the definition of $\boldsymbol{\zeta}(\theta)$, the output feedback control (2.6) can be written as

$$\boldsymbol{u}(\theta) = [K(\theta),\ -K(\theta)]\boldsymbol{\zeta}(\theta) \tag{2.16}$$

In the output feedback control (2.16), the control gain matrix $[K(\theta),\ -K(\theta)]$ can be parameterized as $\mathcal{K}(\theta, \gamma)$, where γ is a parameter to determined. Then, (2.16) becomes a parametric control

$$u(\theta) = \mathcal{K}(\theta, \gamma)\boldsymbol{\zeta}(\theta) \tag{2.17}$$

In the parametric control (2.17), the parameter γ is chosen as the discrete dynamic parameters $\gamma_k, k \in \mathbf{I}[1, N]$. The observer-based output feedback robust H_∞ control (2.17) has the following structure:

$$\begin{cases} u_{k-1}(\theta) = & -\mathcal{K}(\theta, \gamma_{k-1})\boldsymbol{\zeta}(\theta), \hat{Q}(\theta, \gamma_{k-1})\boldsymbol{\zeta}(\theta) \in O(\Lambda_{\max}(\gamma_{k-1}))\backslash O(\Lambda_{\max}(\gamma_k)), \\ & \theta \in [\Theta_{k-1}, \Theta_k), k \in \mathbf{I}[1, N] \\ u_N(\theta) = & -\mathcal{K}(\theta, \gamma_N)\boldsymbol{\zeta}(\theta), \hat{Q}(\theta, \gamma_{k-1})\boldsymbol{\zeta}(\theta) \in O(\Lambda_{\max}(\gamma_N)), \\ & \theta \in [\Theta_N, +\infty) \end{cases} \tag{2.18}$$

where the orthogonal matrix $\hat{Q}(\theta, \gamma_k), k \in \mathbf{I}[0, N]$, and the time-invariant ellipsoids $O(\Lambda_{\max}(\gamma_k)), k \in \mathbf{I}[1, N]$, will be defined later in Appendix 6, respectively.

Now, the observer-based output feedback robust H_∞ control design problem of system (2.15) may now be stated formally as follows: For the given initial state $\boldsymbol{\zeta}(\theta_0)$ of the augmented system (2.15), our object is to design the state observer (2.10) and the observer-based output feedback control (2.18) such that the estimation error of the observer system (2.13) is not only asymptotically stable, but also the spacecraft rendezvous system (2.7) has robust H_∞ performance for external disturbances.

3. Preliminaries

Lemma 3.1. [27] *Let $A \in R^{m\times n}$, $B \in R^{n\times m}$. Then, $\lambda(AB) = \lambda(BA)$, where $\lambda(AB)$ and $\lambda(BA)$ are eigenvalues of AB and BA, respectively.*

Lemma 3.2. [9] *Let $A = (a_{ij}) \in R^{n\times n}$, and $\lambda(A)$ is an eigenvalue of A. Then, $\lambda(A)$ varies continuously as a function of elements a_{ij}, $i, j \in \mathbf{I}[1, n]$.*

Lemma 3.3. *(Cauchy Interlace Theorem in [12]) Let $A \in R^{n\times n}$ be a symmetric matrix, and let $B \in R^{m\times m}$ be a principal sub-matrix of A. Suppose that the eigenvalues of A and B are listed as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$, respectively. Then, $\lambda_{n-m+k} \leq \mu_k \leq \lambda_k, \quad k \in \mathbf{I}[1, m]$.*

Lemma 3.4. *(Courant-Fisher $\min - \max$ Theorem[8]) Let $A \in R^{n\times n}$ be a real symmetric matrix, and let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be all eigenvalues of A. Then,*

$$\lambda_j = \min_{\dim(U)=j} \max_{\boldsymbol{x}\in U, |\boldsymbol{x}|=1} \boldsymbol{x}^\top A \boldsymbol{x}, \quad j \in \mathbf{I}[1, n]$$

Lemma 3.5. [13] *Let X, Y and F be real matrices of appropriate dimensions with $F^\top F \leq I$. Then,*

$$XFY + Y^\top F^\top X^\top \leq \lambda X X^\top + \lambda^{-1} Y^\top Y, \quad \lambda > 0$$

4. Main Results

A discrete gain scheduling control approach has been used in [5] to design the state feedback control of the input saturated elliptical orbit spacecraft rendezvous system. The idea of this discrete gain scheduling control approach is provided in the three steps. For completeness, a brief review is given in Appendix 6. Based on the discrete gain scheduling control approach, observer-based output feedback robust H_∞ control is designed as follows:

Theorem 4.1. *Let $A(\theta), B(\theta)$, and $B_\omega(\theta)$ be defined in system (2.7), $Q_0(\theta, \gamma_i)$ and $R_0(\theta)$ are defined in inequality (2.9). If $P(\theta, \gamma_i)$ is the unique positive definite 2π-periodic solution of periodic Riccati-like differential equation (PRLDE)*

$$\begin{aligned} -P'(\theta) = \quad & A^\mathrm{T}(\theta)P(\theta) + P(\theta)A(\theta) - P(\theta)B(\theta)R_0^{-1}(\theta)B^\top(\theta)P(\theta) + \gamma_i P(\theta) \\ & + \gamma_\omega^{-2} P(\theta)B_\omega(\theta)B_\omega^\top(\theta)P(\theta) + Q_0(\theta, \gamma_i), i \in \mathbf{I}[0, N] \end{aligned} \tag{4.1}$$

and P_e is a positive definite solution of the following matrix inequality

$$\begin{aligned} & [A(\theta) + L(\theta)C(\theta)]^\top P_e + P_e[A(\theta) + L(\theta)C(\theta)] \\ & + \gamma_\omega^{-2}\kappa_0^{-1} P_e B_\omega(\theta) B_\omega^\top(\theta) P_e + Q_e < -I_n, \ \kappa_0 > 0 \end{aligned} \tag{4.2}$$

with inequality constraint $\lambda_{\min}(P_e)T(\gamma_i) < 1$, where

$$T(\gamma_i) = \max_{\theta\in[0,2\pi]} \mathrm{tr}(B^\top(\theta)P(\theta, \gamma_i)B(\theta)R_0^{-1}(\theta))$$

Then, under the following observer-based output feedback discrete gain scheduling control (2.18) the spacecraft rendezvous system (2.7) has robust H_∞ performance for external disturbances. Here,

$$\mathcal{K}(\theta, \gamma_i) = \left[-R_0^{-1}(\theta)B^\top(\theta)P(\theta, \gamma_i), \quad R_0^{-1}(\theta)B^\top(\theta)P(\theta, \gamma_i)\right], i \in \mathbf{I}[0, N] \tag{4.3}$$

γ_N is chosen to satisfy

$$\left\{ \boldsymbol{\zeta}(\theta) : \|\boldsymbol{\zeta}(\theta)\| \leq \gamma_\omega d_\omega \sqrt{\frac{1 + \kappa_0 \delta(\gamma_N)}{\lambda_{\min}(M)}} \right\} \subseteq O(\hat{P}_{\max}(\theta, \gamma_N))$$

where

$$M = \operatorname{diag}\left\{\min_{\theta\in[0,2\pi]} \lambda\{Q_0(\theta,\gamma_0)+\gamma_0 P(\theta,\gamma_0)\}I_n, \delta(\gamma_0)Q_e+\eta_{\min}I_n\right\}$$

and

$$\eta_{\min} = \min_{i\in\mathbf{I}[0,N]}\left\{\delta(\gamma_i)-T(\gamma_i)\max_{\theta\in[0,2\pi]}\lambda(P(\theta,\gamma_i))\right\}>0$$

Proof: Under the control (2.18), system (2.15) can be rewritten as:

$$\begin{cases} \dot{\boldsymbol{\zeta}}(\theta)=\mathcal{A}(\theta)\boldsymbol{\zeta}(\theta)+\mathcal{B}(\theta)\mathrm{sat}(\boldsymbol{u}(\theta))+\mathcal{B}_\omega(\theta)\boldsymbol{\omega}(\theta) \\ \boldsymbol{y}(\theta)=\mathcal{C}(\theta)\boldsymbol{\zeta}(\theta) \end{cases} \tag{4.4}$$

Denote the following set:

$$\Omega_\omega = \{\boldsymbol{\zeta}(\theta)\in R^{12}: ||\mathcal{K}(\theta)\boldsymbol{\zeta}(\theta)||\le\omega\}, \tag{4.5}$$

which implies that the control $u(\theta)=\mathcal{K}(\theta)\boldsymbol{\zeta}(\theta)$ are not saturated if $\boldsymbol{\zeta}(\theta)\in\Omega_\omega$. In the following, we shall show $O(\hat{P}_{\max}(\theta,\gamma_i))\subseteq\Omega_\omega$.

Let

$$\Xi_0(\theta,\gamma_i)=P(\theta,\gamma_i)B(\theta)R_0^{-2}(\theta)B^\top(\theta)P(\theta,\gamma_i)$$

and

$$\Xi_1(\theta,\gamma_i)=P^{\frac{1}{2}}(\theta,\gamma_i)B(\theta)R_0^{-2}(\theta)B^\top(\theta)P^{\frac{1}{2}}(\theta,\gamma_i)$$

then

$$\begin{bmatrix} \Xi_1(\theta,\gamma_i) & -\Xi_1(\theta,\gamma_i) \\ -\Xi_1(\theta,\gamma_i) & \Xi_1(\theta,\gamma_i) \end{bmatrix} \le \begin{bmatrix} 2\Xi_1(\theta,\gamma_i) & 0 \\ 0 & 2\Xi_1(\theta,\gamma_i) \end{bmatrix}$$

which implies

$$\begin{bmatrix} \Xi_1(\theta,\gamma_i) & -\Xi_1(\theta,\gamma_i) \\ -\Xi_1(\theta,\gamma_i) & \Xi_1(\theta,\gamma_i) \end{bmatrix} \le 2\lambda_{\max}\left(\Xi_1(\theta,\gamma_i)I_{12}\right)$$

For $\forall\boldsymbol{\zeta}(\theta)\in O(\hat{P}_{\max}(\theta,\gamma_i))$, we have

$$\begin{aligned} ||\mathcal{K}(\theta)\boldsymbol{\zeta}(\theta)||^2 &= \boldsymbol{\zeta}^\top(\theta)\mathcal{K}^\top(\theta)\mathcal{K}(\theta)\boldsymbol{\zeta}(\theta) \\ &= \boldsymbol{\zeta}^\top(\theta)\begin{bmatrix} \Xi_0(\theta,\gamma_i) & -\Xi_0(\theta,\gamma_i) \\ -\Xi_0(\theta,\gamma_i) & \Xi_0(\theta,\gamma_i) \end{bmatrix}\boldsymbol{\zeta}(\theta) \\ &= \boldsymbol{\zeta}^\top(\theta)\mathrm{diag}\{P^{\frac{1}{2}}(\theta,\gamma_i),P^{\frac{1}{2}}(\theta,\gamma_i)\}\begin{bmatrix} \Xi_1(\theta,\gamma_i) & -\Xi_1(\theta,\gamma_i) \\ -\Xi_1(\theta,\gamma_i) & \Xi_1(\theta,\gamma_i) \end{bmatrix} \\ \times\mathrm{diag}\{P^{\frac{1}{2}}(\theta,\gamma_i),P^{\frac{1}{2}}(\theta,\gamma_i)\}\boldsymbol{\zeta}(\theta) &\le 2\lambda_{\max}\big(\Xi_1(\theta,\gamma_i)\big)\boldsymbol{\zeta}^\top(\theta)\mathrm{diag}\{P(\theta,\gamma_i),P(\theta,\gamma_i)\}\boldsymbol{\zeta}(\theta) \end{aligned} \tag{4.6}$$

Let

$$\Xi_2(\theta,\gamma_i)=R_0^{-1}(\theta)B^\top(\theta)P(\theta,\gamma_i)B(\theta)R_0^{-1}(\theta)$$

From Lemma 3.1, it follows

$$\lambda_{\max}(\Xi_1(\theta,\gamma_i))=\lambda_{\max}(\Xi_2(\theta,\gamma_i))$$

Thus,

$$||\mathcal{K}(\theta)\boldsymbol{\zeta}(\theta)||^2\le 2\lambda_{\max}\big(\Xi_2(\theta,\gamma_i)\big)\boldsymbol{\zeta}^\top(\theta)\mathrm{diag}\{P(\theta,\gamma_i),P(\theta,\gamma_i)\}\boldsymbol{\zeta}(\theta) \tag{4.7}$$

Let $P(\theta,\gamma_i)$ be the solution of the PRDE (4.1) with the parameter γ_i, $i\in\mathbf{I}[1,N]$, which is divided as a block matrix given below:

$$P(\theta,\gamma_i)=\begin{bmatrix} P_{11}(\theta,\gamma_i) & P_{12}(\theta,\gamma_i) \\ P_{12}^T(\theta,\gamma_i) & P_{22}(\theta,\gamma_i) \end{bmatrix}$$

According to the specific forms of $B(\theta)$ in (2.5a) and $R_0(\theta)$ in (2.9), it is easy to compute

$$\begin{aligned} \Xi_2(\theta,\gamma_i) = &\ R_0^{-1}(\theta)B^\top(\theta)P(\theta,\gamma_i)B(\theta)R_0^{-1}(\theta) \\ = &\ r^{-2}(\theta)k^{-8}\rho^{-6}(\theta)P_{22}(\theta,\gamma_i) \end{aligned}$$

From Lemma 3.3, it follows that $\lambda_{\max}(P_{22}(\theta,\gamma_i)) \leq \lambda_{\max}(P(\theta,\gamma_i))$, meaning that

$$\begin{aligned}\lambda_{\max}(\Xi_2(\theta,\gamma_i)) \leq \;& \lambda_{\max}(r^{-2}(\theta)k^{-8}\rho^{-6}(\theta)P(\theta,\gamma_i)) \\ \leq \;& 2^{-1}\hat{k}\hat{\lambda}_1(\gamma_i)\end{aligned} \tag{4.8}$$

where $\hat{\lambda}_1(\gamma_i)$ is the maximum eigenvalue of $P(\theta,\gamma_i)$ on $[0,2\pi]$, and

$$\hat{k} = 2\hat{r}^{-2}k^{-8}\hat{\rho}^{-6}, \quad \hat{r} = \max_{\theta\in[0,2\pi]} r(\theta), \quad \hat{\rho} = \max_{\theta\in[0,2\pi]} \rho(\theta)$$

Combining (4.7) with (4.8), we have

$$\begin{aligned}||\mathcal{K}(\theta)\boldsymbol{\zeta}(\theta)||^2 &\leq \hat{k}\boldsymbol{\zeta}^\top(\theta)\text{diag}\{\hat{\lambda}_1(\gamma_i)\hat{P}(\theta,\gamma_i), \hat{\lambda}_1^2(\gamma_i)\lambda_{\min}^{-1}(P_e)P_e\}\boldsymbol{\zeta}(\theta) \\ &\leq \boldsymbol{\zeta}^\top(\theta)\hat{P}_{\max}(\theta,\gamma_i)\boldsymbol{\zeta}(\theta) \\ &\leq \omega^2\end{aligned}$$

This inequality implies $O_\omega(\hat{P}_{\max}(\theta,\gamma_i)) \subseteq \Omega_\omega$, which means that the control of the augmented system (2.15) is not saturated for any $\boldsymbol{\zeta}(\theta) \in O_\omega(\hat{P}_{\max}(\theta,\gamma_i))$, i.e., $\text{sat}(\mathcal{K}(\theta,\gamma_i)\boldsymbol{\zeta}(\theta)) = \mathcal{K}(\theta,\gamma_i)\boldsymbol{\zeta}(\theta)$.

Construct the following Lyapunov candidate function

$$\begin{aligned}V_i(\theta,\boldsymbol{\zeta}(\theta)) &= \boldsymbol{\zeta}^\top(\theta)\text{diag}\{P(\theta,\gamma_i), \delta(\gamma_i)P_e\}\boldsymbol{\zeta}(\theta) \\ &= \boldsymbol{x}^\top(\theta)P(\theta,\gamma_i)\boldsymbol{x}(\theta) + \delta(\gamma_i)\boldsymbol{e}^\top(\theta)P_e\boldsymbol{e}(\theta)\end{aligned} \tag{4.9}$$

where $\delta(\gamma_i) = \hat{\lambda}_1(\gamma_i)\lambda_{\min}^{-1}(P_e)$. Take derivative on both sides of (4.9) with respect to θ, we have

$$\begin{aligned}V_i'(\theta,\boldsymbol{\zeta}(\theta)) = \;& \boldsymbol{x}^\top(\theta)[A^\top(\theta)P(\theta,\gamma_i) + P(\theta,\gamma_i)A(\theta) + P'(\theta,\gamma_i)]\boldsymbol{x}(\theta) \\ & +2\boldsymbol{x}^\top(\theta)P(\theta,\gamma_i)B(\theta)\text{sat}(\boldsymbol{u}(\theta)) + 2\boldsymbol{x}^\top(\theta)P(\theta,\gamma_i)B_\omega(\theta)\boldsymbol{\omega}(\theta) \\ & +\delta(\gamma_i)\boldsymbol{e}^\top(\theta)\{[A(\theta)+L(\theta)C(\theta)]^\top P_e + P_e[A(\theta)+L(\theta)C(\theta)]\}\boldsymbol{e}(\theta) \\ & +2\delta(\gamma_i)\boldsymbol{e}^\top(\theta)P_eB_\omega(\theta)\boldsymbol{\omega}(\theta)\end{aligned} \tag{4.10}$$

Let

$$\begin{aligned}\Phi_i = \;& V_i'(\theta,\boldsymbol{\zeta}(\theta)) + \boldsymbol{z}^\top(\theta)\boldsymbol{z}(\theta) + \delta(\gamma_i)\boldsymbol{e}^\top(\theta)Q_e\boldsymbol{e}(\theta) \\ & -(1+\kappa_0\delta(\gamma_i))\gamma_\omega^2\boldsymbol{\omega}^\top(\theta)\boldsymbol{\omega}(\theta)\end{aligned} \tag{4.11}$$

From (2.8), it is easy to deduce

$$\begin{aligned}\boldsymbol{z}^\top(\theta)\boldsymbol{z}(\theta) = \;& \boldsymbol{x}^\top(\theta)C_0^\top(\theta,\gamma_i)C_0(\theta,\gamma_i)\boldsymbol{x}(\theta) + \boldsymbol{u}^\top(\theta)D_0^\top(\theta)D_0(\theta)\boldsymbol{u}(\theta) \\ = \;& \boldsymbol{x}^\top(\theta)Q_0(\theta,\gamma_i)\boldsymbol{x}(\theta) + \boldsymbol{u}^\top(\theta)R_0(\theta)\boldsymbol{u}(\theta)\end{aligned} \tag{4.12}$$

where $Q_0(\theta,\gamma_i)$ and $R_0(\theta)$ are defined in (2.9). According to (4.10) and (4.12), then Φ_i in (4.11) is in the form as follows

$$\begin{aligned}\Phi_i = \;& \boldsymbol{x}^\top(\theta)[A^\top(\theta)P(\theta,\gamma_i) + P(\theta,\gamma_i)A(\theta) + P'(\theta,\gamma_i)]\boldsymbol{x}(\theta) \\ & +2\boldsymbol{x}^\top(\theta)P(\theta,\gamma_i)B(\theta)\text{sat}(\boldsymbol{u}_i(\theta)) + 2\boldsymbol{x}^\top(\theta)P(\theta,\gamma_i)B_\omega(\theta)\boldsymbol{\omega}(\theta) \\ & +\delta(\gamma_i)\boldsymbol{e}^\top(\theta)\{[A(\theta)+L(\theta)C(\theta)]^\top P_e + P_e[A(\theta)+L(\theta)C(\theta)]\}\boldsymbol{e}(\theta) \\ & +2\delta(\gamma_i)\boldsymbol{e}^\top(\theta)P_eB_\omega(\theta)\boldsymbol{\omega}(\theta) + \boldsymbol{x}^\top(\theta)Q_0(\theta,\gamma_i)\boldsymbol{x}(\theta) + \boldsymbol{u}_i^\top(\theta)R_0(\theta)\boldsymbol{u}_i(\theta) \\ & +\delta(\gamma_i)\boldsymbol{e}^\top(\theta)Q_e\boldsymbol{e}(\theta) - (1+\kappa_0\delta(\gamma_i))\gamma_\omega^2\boldsymbol{\omega}^\top(\theta)\boldsymbol{\omega}(\theta)\end{aligned} \tag{4.13}$$

By using Lemma 3.5, we can obtain

$$\begin{aligned}\Phi_i \leq \;& \boldsymbol{x}^\top(\theta)[A^\top(\theta)P(\theta,\gamma_i) + P(\theta,\gamma_i)A(\theta) + P'(\theta,\gamma_i)]\boldsymbol{x}(\theta) \\ & +\boldsymbol{x}^\top(\theta)[\gamma_\omega^{-2}P(\theta,\gamma_i)B_\omega(\theta)B_\omega^\top(\theta)P(\theta,\gamma_i) + Q_0(\theta,\gamma_i)]\boldsymbol{x}(\theta) \\ & +\delta(\gamma_i)\boldsymbol{e}^\top(\theta)\{[A(\theta)+L(\theta)C(\theta)]^\top P_e + P_e[A(\theta)+L(\theta)C(\theta)]\}\boldsymbol{e}(\theta) \\ & +\delta(\gamma_i)\gamma_\omega^{-2}\kappa_0^{-1}\boldsymbol{e}^\top(\theta)P_eB_\omega(\theta)B_\omega^\top(\theta)P_e\boldsymbol{e}(\theta) + \delta(\gamma_i)\kappa_0\gamma_\omega^2\boldsymbol{\omega}^\top(\theta)\boldsymbol{\omega}(\theta) \\ & +\boldsymbol{u}_i^\top(\theta)R_0(\theta)\boldsymbol{u}_i(\theta) + \delta(\gamma_i)\boldsymbol{e}^\top(\theta)Q_e\boldsymbol{e}(\theta) + \gamma_\omega^2\boldsymbol{\omega}^\top(\theta)\boldsymbol{\omega}(\theta) \\ & +2\boldsymbol{x}^\top(\theta)P(\theta,\gamma_i)B(\theta)\text{sat}(\boldsymbol{u}_i(\theta)) - (1+\kappa_0\delta(\gamma_i))\gamma_\omega^2\boldsymbol{\omega}^\top(\theta)\boldsymbol{\omega}(\theta)\end{aligned} \tag{4.14}$$

i.e.,

$$\begin{array}{rl}\Phi_i \leq & \boldsymbol{x}^\top(\theta)[A^\top(\theta)P(\theta,\gamma_i)+P(\theta,\gamma_i)A(\theta)+P'(\theta,\gamma_i)]\boldsymbol{x}(\theta)\\ & +\boldsymbol{x}^\top(\theta)[\gamma_\omega^{-2}P(\theta,\gamma_i)B_\omega(\theta)B_\omega^\top(\theta)P(\theta,\gamma_i)+Q_0(\theta,\gamma_i)]\boldsymbol{x}(\theta)\\ & +\delta(\gamma_i)\boldsymbol{e}^\top(\theta)\left\{[A(\theta)+L(\theta)C(\theta)]^\top P_e+P_e[A(\theta)+L(\theta)C(\theta)]\right\}\boldsymbol{e}(\theta)\\ & +2\boldsymbol{x}^\top(\theta)P(\theta,\gamma_i)B(\theta)\mathrm{sat}(\boldsymbol{u}_i(\theta))+\boldsymbol{u}_i^\top(\theta)R_0(\theta)\boldsymbol{u}_i(\theta)\\ & +\delta(\gamma_i)\gamma_\omega^{-2}\kappa_0^{-1}\boldsymbol{e}^\top(\theta)P_eB_\omega(\theta)B_\omega^\top(\theta)P_e\boldsymbol{e}(\theta)+\delta(\gamma_i)\boldsymbol{e}^\top(\theta)Q_e\boldsymbol{e}(\theta)\end{array} \tag{4.15}$$

In view of $\boldsymbol{x}(\theta)=\boldsymbol{e}(\theta)+\bar{\boldsymbol{x}}(\theta)$, (4.15) can be computed as

$$\begin{array}{rl}\Phi_i \leq & \boldsymbol{x}^\top(\theta)[A^\top(\theta)P(\theta,\gamma_i)+P(\theta,\gamma_i)A(\theta)+P'(\theta,\gamma_i)]\boldsymbol{x}^\top(\theta)\\ & +\boldsymbol{x}^\top(\theta)[\gamma_\omega^{-2}P(\theta,\gamma_i)B_\omega(\theta)B_\omega^\top(\theta)P(\theta,\gamma_i)+Q_0(\theta,\gamma_i)]\boldsymbol{x}(\theta)\\ & +\delta(\gamma_i)\boldsymbol{e}^\top(\theta)\{[A(\theta)+L(\theta)C(\theta)]^\top P_e+P_e[A(\theta)+L(\theta)C(\theta)]\boldsymbol{e}(\theta)\\ & +2\bar{\boldsymbol{x}}^\top(\theta)P(\theta,\gamma_i)B(\theta)\mathrm{sat}(\boldsymbol{u}_i(\theta))+\boldsymbol{u}_i^\top(\theta)R_0(\theta)\boldsymbol{u}_i(\theta)+\boldsymbol{e}^\top(\theta)Q_e\boldsymbol{e}(\theta)\\ & +2\boldsymbol{e}^\top(\theta)P(\theta,\gamma_i)B(\theta)\mathrm{sat}(\boldsymbol{u}_i(\theta))+\delta(\gamma_i)\gamma_\omega^{-2}\kappa_0^{-1}\boldsymbol{e}^\top(\theta)P_eB_\omega(\theta)B_\omega^\top(\theta)P_e\boldsymbol{e}(\theta)\end{array} \tag{4.16}$$

Since $\boldsymbol{\zeta}(\theta)\in O_\omega(\hat{P}_{\max}(\theta,\gamma_i))$, then $\mathrm{sat}(\boldsymbol{u}_i(\theta))$ is not saturated, i.e., $\mathrm{sat}(\boldsymbol{u}_i(\theta))=\boldsymbol{u}_i(\theta)$, it follows from (4.16) that

$$\begin{array}{rl}\Phi_i \leq & \boldsymbol{x}^\top(\theta)[A^\top(\theta)P(\theta,\gamma_i)+P(\theta,\gamma_i)A(\theta)+P'(\theta,\gamma_i)]\boldsymbol{x}(\theta)\\ & +\boldsymbol{x}^\top(\theta)[\gamma_\omega^{-2}P(\theta,\gamma_i)B_\omega(\theta)B_\omega^\top(\theta)P(\theta,\gamma_i)+Q_0(\theta,\gamma_i)]\boldsymbol{x}(\theta)\\ & +[\boldsymbol{u}_i(\theta)+R_0^{-1}(\theta)B^\top(\theta)P(\theta,\gamma_i)\bar{\boldsymbol{x}}(\theta)]^\top R_0(\theta)\\ & \cdot[\boldsymbol{u}_i(\theta)+R_0^{-1}(\theta)B^\top(\theta)P(\theta,\gamma_i)\bar{\boldsymbol{x}}(\theta)]\\ & -\bar{\boldsymbol{x}}^\top(\theta)P(\theta,\gamma_i)B(\theta)R_0^{-1}B^\top(\theta)P(\theta,\gamma_i)\bar{\boldsymbol{x}}(\theta)+2\boldsymbol{e}^\top(\theta)P(\theta,\gamma_i)B(\theta)\boldsymbol{u}_i(\theta)\\ & +\delta(\gamma_i)\boldsymbol{e}^\top(\theta)[A(\theta)+L(\theta)C(\theta)]^\top P_e+P_e[A(\theta)+L(\theta)C(\theta)]\boldsymbol{e}(\theta)\\ & +\delta(\gamma_i)\boldsymbol{e}^\top(\theta)[\gamma_\omega^{-2}\kappa_0P_eB_\omega(\theta)B_\omega^\top(\theta)P_e+Q_e]\boldsymbol{e}(\theta)\end{array} \tag{4.17}$$

Substituting

$$\boldsymbol{u}_i(\theta)=-R_0^{-1}(\theta)B^\top(\theta)P(\theta)\bar{\boldsymbol{x}}(\theta),\quad \bar{\boldsymbol{x}}^\top(\theta)=\boldsymbol{x}^\top(\theta)-\boldsymbol{e}^\top(\theta) \tag{4.18}$$

into the right side of inequality (4.17), yields

$$\begin{array}{rl}\Phi_i \leq & \boldsymbol{x}^\top(\theta)[A^\top(\theta)P(\theta,\gamma_i)+P(\theta,\gamma_i)A(\theta)+P'(\theta,\gamma_i)]\boldsymbol{x}(\theta)\\ & +\boldsymbol{x}^\top(\theta)[\gamma_\omega^{-2}P(\theta,\gamma_i)B_\omega(\theta)B_\omega^\top(\theta)P(\theta,\gamma_i)+Q_0(\theta,\gamma_i)]\boldsymbol{x}(\theta)\\ & -\boldsymbol{x}^\top(\theta)P(\theta,\gamma_i)B(\theta)R_0^{-1}(\theta)B^\top(\theta)P(\theta,\gamma_i)\boldsymbol{x}(\theta)\\ & +\boldsymbol{e}^\top(\theta)P(\theta,\gamma_i)B(\theta)R_0^{-1}(\theta)B^\top(\theta)P(\theta,\gamma_i)\boldsymbol{e}(\theta)\\ & +\delta(\gamma_i)\boldsymbol{e}^\top(\theta)[A(\theta)+L(\theta)C(\theta)]^\top P_e\boldsymbol{e}(\theta)\\ & +\delta(\gamma_i)\boldsymbol{e}^\top(\theta)[P_e[A(\theta)+L(\theta)C(\theta)]\boldsymbol{e}(\theta)\\ & +\boldsymbol{e}^\top(\theta)[\delta(\gamma_i)\gamma_\omega^{-2}\kappa_0^{-1}P_eB_\omega(\theta)B_\omega^\top(\theta)P_e+\delta(\gamma_i)Q_e]\boldsymbol{e}(\theta)\end{array} \tag{4.19}$$

Using PRDE (4.1) and matrix inequality (4.2), (4.19) can be simplified as

$$\begin{array}{rl}\Phi_i \leq & -\gamma_i\boldsymbol{x}^\top(\theta)P(\theta,\gamma_i)\boldsymbol{x}^\top(\theta)-\delta(\gamma_i)\boldsymbol{e}^\top(\theta)\boldsymbol{e}(\theta)\\ & +\mathrm{tr}(B^\top(\theta)P(\theta,\gamma_i)B(\theta)R_0^{-1}(\theta))\boldsymbol{e}^\top(\theta)P(\theta,\gamma_i)\boldsymbol{e}(\theta)\\ \leq & -\gamma_i\boldsymbol{x}^\top(\theta)P(\theta,\gamma_i)\boldsymbol{x}^\top(\theta)-(\lambda_{\min}^{-1}(P_e)-T(\gamma_i))\hat{\lambda}_1(\gamma_i)\boldsymbol{e}^\top(\theta)\boldsymbol{e}(\theta)\end{array} \tag{4.20}$$

It is clear that $\Phi_i\leq 0$. Thus,

$$\begin{array}{rl}V_i'(\theta,\boldsymbol{\zeta}(\theta)) \leq & -\boldsymbol{u}^\top(\theta)R_0(\theta)\boldsymbol{u}(\theta)-\boldsymbol{x}^\top(\theta)Q_0(\theta,\gamma_i)\boldsymbol{x}(\theta)\\ & -\gamma_i\boldsymbol{x}^\top(\theta)P(\theta,\gamma_i)\boldsymbol{x}^\top(\theta)-\delta(\gamma_i)\boldsymbol{e}^\top(\theta)Q_e\boldsymbol{e}(\theta)\\ & -\eta_i\boldsymbol{e}^\top(\theta)\boldsymbol{e}(\theta)+(1+\kappa_0\delta(\gamma_i))\gamma_\omega^2\boldsymbol{\omega}^\top(\theta)\boldsymbol{\omega}(\theta)\\ \leq & -\bar{\boldsymbol{x}}^\top(\theta)P(\theta,\gamma_i)B(\theta)R_0^{-1}(\theta)B^\top(\theta)P(\theta)\bar{\boldsymbol{x}}(\theta)\\ & -\boldsymbol{\zeta}^\top(\theta)\mathrm{diag}\{Q_0(\theta,\gamma_i)+\gamma_iP(\theta,\gamma_i),\delta(\gamma_i)Q_e+\eta_iI_n\}\boldsymbol{\zeta}(\theta)\\ & +(1+\kappa_0\delta(\gamma_i))\gamma_\omega^2\boldsymbol{\omega}^\top(\theta)\boldsymbol{\omega}(\theta)\\ \leq & -\bar{\boldsymbol{x}}^\top(\theta)P(\theta,\gamma_i)B(\theta)R_0^{-1}(\theta)B^\top(\theta)P(\theta,\gamma_i)\bar{\boldsymbol{x}}(\theta)\end{array} \tag{4.21}$$

where

$$\|\boldsymbol{\zeta}(\theta)\| \geq \gamma_\omega d_\omega \sqrt{\frac{1+\kappa_0\delta(\gamma_N)}{\lambda_{\min}(M)}}$$

$$M = \quad \operatorname{diag}\left\{\min_{\theta\in[0,2\pi]} \lambda\{Q_0(\theta,\gamma_0)+\gamma_0 P(\theta,\gamma_0)\}I_n, \delta(\gamma_0)Q_e+\eta_{\min}I_n\right\}$$

and

$$\eta_{\min} = \min_{i\in\mathbf{I}[0,N]}[\delta(\gamma_i)-T(\gamma_i)\hat{\lambda}_1(\gamma_i)] > 0$$

By using the Lyapunov stability theory, the states of the augmented system (2.14) will move into the region $O(\hat{P}_{\max}(\theta,\gamma_N))$ and eventually converge to $O(\hat{P}_{\max}(\theta,\gamma_N))$ and stay in the region $O(\hat{P}_{\max}(\theta,\gamma_N))$ all the time.

From (4.20), it is easy to obtain

$$\Phi_N \leq -\gamma_N \min_{\theta\in[0,2\pi]} \lambda(P(\theta,\gamma_N))\boldsymbol{x}^\top(\theta)\boldsymbol{x}^\top(\theta) - \eta_N \boldsymbol{e}^\top(\theta)\boldsymbol{e}(\theta) \tag{4.22}$$

When the external disturbance $\omega = 0$, from (4.22), it is easy to obtain

$$V_N'(\theta,\boldsymbol{\zeta}(\theta)) < -\boldsymbol{z}^\top(\theta)\boldsymbol{z}(\theta) - \delta(\gamma_N)\boldsymbol{e}^\top(\theta)Q_e\boldsymbol{e}(\theta)$$

Based on the LaSalles invariant set theorem, we can conclude that the closed-loop system (2.15) is asymptotically stable.

When the external disturbance $\omega \neq 0$, from (4.22), it is also easy to obtain $\Phi_N < 0$. Integrating both sides of the inequality $\Phi_N < 0$ from 0 to $+\infty$, we have

$$\begin{aligned}&\int_0^{+\infty} V_N'(\theta,\boldsymbol{\zeta}(\theta)) + \boldsymbol{z}^\top(\theta)\boldsymbol{z}(\theta) + \delta(\gamma_N)\boldsymbol{e}^\top(\theta)Q_e\boldsymbol{e}(\theta)d\theta \\ &-\int_0^{+\infty}(1+\kappa_0\delta(\gamma_N))\gamma_\omega^2\boldsymbol{\omega}^\top(\theta)\boldsymbol{\omega}(\theta)d\theta < 0\end{aligned} \tag{4.23}$$

i.e.,

$$\begin{aligned}&V_N(+\infty,\boldsymbol{\zeta}(+\infty)) - V_N(0,\boldsymbol{\zeta}(0)) \\ &+\int_0^{+\infty}\boldsymbol{x}^\top(\theta)Q_0(\theta,\gamma_N)\boldsymbol{x}(\theta)+\boldsymbol{u}^\top(\theta)R_0(\theta)\boldsymbol{u}(\theta)d\theta \\ &+\int_0^{+\infty}\delta(\gamma_N)\boldsymbol{e}^\top(\theta)Q_e\boldsymbol{e}(\theta) - (1+\kappa_0\delta(\gamma_N))\gamma_\omega^2\boldsymbol{\omega}^\top(\theta)\boldsymbol{\omega}(\theta)d\theta < 0\end{aligned} \tag{4.24}$$

From the initial condition $\boldsymbol{\zeta}(0) = 0$, it follows that

$$\begin{aligned}&\int_0^{+\infty}\boldsymbol{x}^\top(\theta)Q_0(\theta,\gamma_N)\boldsymbol{x}(\theta)+\boldsymbol{u}^\top(\theta)R_0(\theta)\boldsymbol{u}(\theta)d\theta \\ &+\int_0^{+\infty}\delta(\gamma_N)\boldsymbol{e}^\top(\theta)Q_e\boldsymbol{e}(\theta) < (1+\kappa_0\delta(\gamma_N))\gamma_\omega^2\boldsymbol{\omega}^\top(\theta)\boldsymbol{\omega}(\theta)d\theta\end{aligned} \tag{4.25}$$

which can be simplified as

$$\begin{aligned}&\int_0^{+\infty}\boldsymbol{x}^\top(\theta)Q_0(\theta,\gamma_N)\boldsymbol{x}(\theta)+\boldsymbol{u}^\top(\theta)R_0(\theta)\boldsymbol{u}(\theta)d\theta \\ &\leq (1+\kappa_0\delta(\gamma_N))\int_0^{+\infty}\gamma_\omega^2\boldsymbol{\omega}^\top(\theta)\boldsymbol{\omega}(\theta)d\theta\end{aligned} \tag{4.26}$$

This implies spacecraft rendezvous system (2.7) with the observer-based output feedback discrete gain scheduling control (2.18) has robust H_∞ performance.

Remark 4.2. Since $(A(\theta), C(\theta))$ is observable, there exists a periodic matrix $L(\theta)$ such that $A(\theta) + L(\theta)C(\theta)$ is a constant Hurwitz matrix. Based on the Lyapunov stability theory, there exists a positive definite matrix P_e such that $(A(\theta) + L(\theta)C(\theta))^\top P_e + P_e(A(\theta) + L(\theta)C(\theta)) < 0$. By setting the appropriate parameter κ_0, the positive definite matrix P_e can be obtained by solving the inequality (4.2). In addition, to ensure that the obtained positive definite matrix P_e satisfies that the inequality constraint condition $\lambda_{\min}(P_e)T(\gamma_i) < 1$ in the following proof of Theorem 4.1, an extra linear matrix inequality $P_e < \alpha I_n$ should be added, where α is a small positive constant.

Remark 4.3. For PRLDE (4.1), if the system (2.15) is not affected by external disturbances $\boldsymbol{\omega}(\theta)$ or the system (2.15) is affected by external disturbances $\boldsymbol{\omega}(\theta)$, and $\gamma_\omega^2 I_n > R_0(\theta)$, then periodic Riccati-like differential equation (4.1) can be simplified as periodic Riccati differential equation

$$\begin{aligned} P'(\theta) = \; & \left(A(\theta) + \frac{1}{2}\gamma_i I_6\right)^{\mathrm{T}} P(\theta) + P(\theta)\left(A(\theta) + \frac{1}{2}\gamma_i I_6\right) \\ & - P(\theta)B(\theta)R_0^{-1}(\theta)B^\top(\theta)P(\theta) + C_0^\top(\theta,\gamma_i)C_0(\theta,\gamma_i), \\ & i \in \mathbf{I}[0,N] \end{aligned} \tag{4.27a}$$

$$\begin{aligned} P'(\theta) = \; & \left(A(\theta) + \frac{1}{2}\gamma_i I_6\right)^{\mathrm{T}} P(\theta) + P(\theta)\left(A(\theta) + \frac{1}{2}\gamma_i I_6\right) \\ & - P(\theta)B(\theta)(R_0^{-1}(\theta) - \gamma_\omega^{-2} I_6)B^\top(\theta)P(\theta) + C_0^\top(\theta,\gamma_i)C_0(\theta,\gamma_i), \\ & i \in \mathbf{I}[0,N] \end{aligned} \tag{4.27b}$$

Since matrix pair $[A(\theta), B(\theta)]$ is controllable, matrix pair $[A(\theta), C_0(\theta)]$ is observable, then

$$\left[A(\theta) + \frac{1}{2}\gamma_i I_6, B(\theta)\right], \quad \left[A(\theta) + \frac{1}{2}\gamma_i I_6, C_0(\theta)\right]$$

is controllable and observable, respectively. Thus, (4.27a) and (4.27b) have both unique positive definite solutions, which have been shown from [1]. Based on the above analysis, our object is find a minimum $\gamma_\omega > 0$ such that periodic Riccati-like differential equation (4.1) has a unique positive definite solution, which can be verified in Numerical Simulations 5.

Remark 4.4. Based on Theorem 4.1, we have the following algorithm, referred to as Algorithm A, for the design of observer-based output feedback robust H_∞ control.

Step 1. Solving PRDE (4.27a) by using Fourier expansion based recursive algorithms in [15].

Step 2. The obtained solution of PRDE from Step 1 is chosen as the initial condition, a new iterative algorithm is applied in [4] to solve periodic Riccati-like differential equations (4.1).

Step 3. Using the periodic solution of PRLDE (4.1) from Step 2, observer-based output feedback robust H_∞ control (2.18) can be obtained, whose switching points can be determined by algorithm A in [5].

5. Numerical Simulations

In this section, a practical example is given to verify advantages of the proposed control design method in this paper.

Suppose that the target spacecraft is in a geostationary transfer orbit, and the chaser spacecraft is in its neighborhood, i.e., the distance between the target spacecraft and the chaser spacecraft is much smaller than the distance between the target spacecraft and the center of the earth. For clarity, the main parameters of the target spacecraft are listed in Table 1 of [30].

For the purpose of the numerical simulation, the initial condition of system (2.2) in the target-orbital coordinate system is chosen as $\bar{\zeta}(t_0) = [2400, -2400, 2400, -5, -2, -2]^\top$, i.e., the relative distances $x(t_0), y(t_0)$ and $z(t_0)$ between the two spacecrafts in the direction of three coordinate axes are 2400m,

-2400m and 2400m, respectively, and their corresponding relative velocities $\dot{x}(t_0), \dot{y}(t_0)$ and $\dot{z}(t_0)$, are -5 m/s, -2 m/s and -2 m/s, respectively. Let the initial true anomaly be $\theta_0 = 0.1\pi$, which is corresponding to the initial time t_0. By using the linear transformation (2.3), the initial condition for linear periodic system (2.4) can be computed as

$$x(\theta_0) = [4068 - 40684068 - 6283 - 1754 - 2838]]^\top \tag{5.1}$$
$$\omega(t) = [\sin(t), \sin(t), \sin(t)]^\top$$

The initial condition for the observer error system (2.4) is

$$e(\theta_0) = [10, 10, 10, 1, 1, 1]^\top \tag{5.2}$$

In equation (4.27b), take $\gamma_\omega^{-2} = 1$, and $R_0^{-1}(\theta) = 2I_6$. Combining (5.1) with (5.2), the initial condition for the augmented system (2.15) is as follows

$$\xi(\theta_0) = [x^\top(\theta_0), e^\top(\theta_0)]^\top$$

Choose the observer gain matrix

$$L(\theta) = \begin{bmatrix} -0.9 & 0 & 2.5 \\ 0.5 & -1 & -0.5 \\ -0.85 & 0.2 & -1 \\ -\frac{3}{\rho(\theta)} & 0.1 & -2.4 \\ 0 & -2.55 & -1.5 \\ 0 & -0.5 & -1 \end{bmatrix} \tag{5.3}$$

such that $A(\theta) + L(\theta)C(\theta)$ is a constant Hurwitz matrix. This means the inequality (2.14) has the unique positive definitive solution

$$P_e = \begin{bmatrix} 1.3623 & -0.8602 & -1.3602 & 0.4857 & 1.2495 & -0.2717 \\ -0.8602 & 3.7012 & 0.2482 & 0.5357 & -1.4857 & 1.3813 \\ -1.3602 & 0.2482 & 4.7621 & -1.5365 & -2.0660 & -0.5000 \\ 0.4857 & 0.5357 & -1.5365 & 1.5962 & 0.4929 & 0.3423 \\ 1.2495 & -1.4857 & -2.0660 & 0.4929 & 2.4888 & -0.9041 \\ -0.2717 & 1.3813 & -0.5000 & 0.3423 & -0.9041 & 2.2134 \end{bmatrix} \tag{5.4}$$

In order to validate that the discrete gain scheduling control (2.18) can improve the dynamic performance of the system with the increase of the swiching point number N. We provide three different cases: $N = 0, N = 10$ and $N = 30$. For the given initial parameter $\gamma_0 > 0$, the PRLDE 4.1 has the unique positive definite solution $P(\theta_0, \gamma_0)$, and there exists an orthogonal decomposition such that

$$P(\theta_0, \gamma_0) = Q^\top(\theta_0, \gamma_0)\Lambda(\theta_0, \gamma_0)Q(\theta_0, \gamma_0)$$

where $Q(\theta_0, \gamma_0)$ is an orthogonal matrix. According to Step 1 of Algorithm A in [5], a feasible initial parameter $\gamma_0 = 0.85$ can be obtained by solving the nonlinear equation

$$[Q(\theta_0, \gamma)\xi(\theta_0)]^\top \Lambda_{\max}(\gamma)[Q(\theta_0, \gamma)\xi(\theta_0)] = 0.9738 \tag{5.5}$$

where $\omega^2 = 0.9738$. Choose $\Delta\gamma = 1.05$, the parameter set Γ_N can be determined by using Step 2 of Algorithm A in [5]. From the parameter set Γ_N, the switching points can be computed according to Step 3 of Algorithm A in [5]. Based on Step 2 and Step 3 of Algorithm A in [5], the discrete gain scheduling control (2.18) can easily be obtained.

Under the obtained discrete gain scheduling control (2.18), the state response signals of the closed-loop spacecraft rendezvous and docking system are shown in Figures 2 to 7. As can be seen from Figures 2 to 7, the state of the system gradually approaches zero after a period of time, which means that the target spacecraft and the tracking spacecraft can finally rendezvous, and the convergence speed of the system becomes faster and faster with the increase of the number of switching points, which indicates that discrete gain scheduling control can improve the dynamic performance of spacecraft rendezvous. The control input signals for the system are shown in Figures 8 to 10. As can be seen from Figures 8 to 10, the proposed controller can make full use of the control capability, but the control input does not exceed the maximum allowable constraint during the whole rendezvous process, which indicates that the discrete gain scheduling control designed in this paper can not only realize the spacecraft rendezvous, but also meet the practical constraints of spacecraft control.

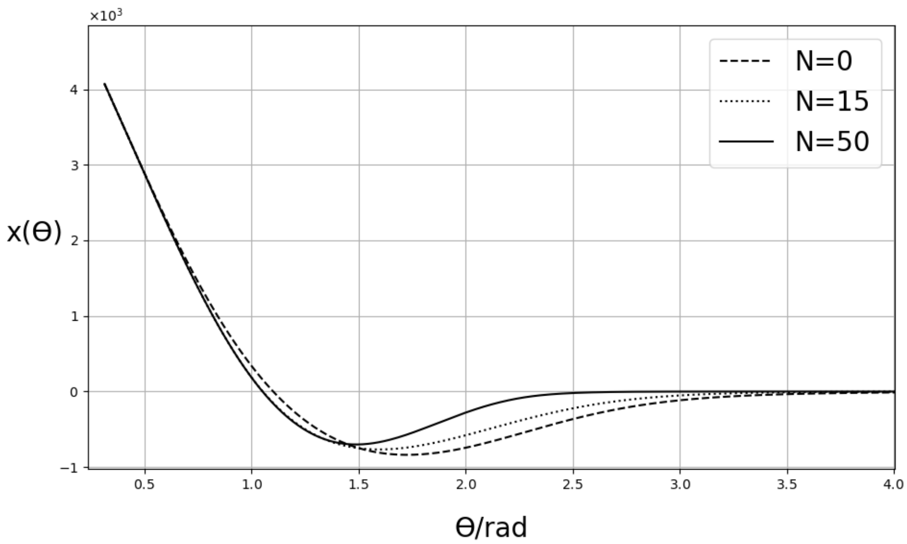

FIGURE 2. The position trajectories in x-axis direction with different switching points for N = 0, 15, 50

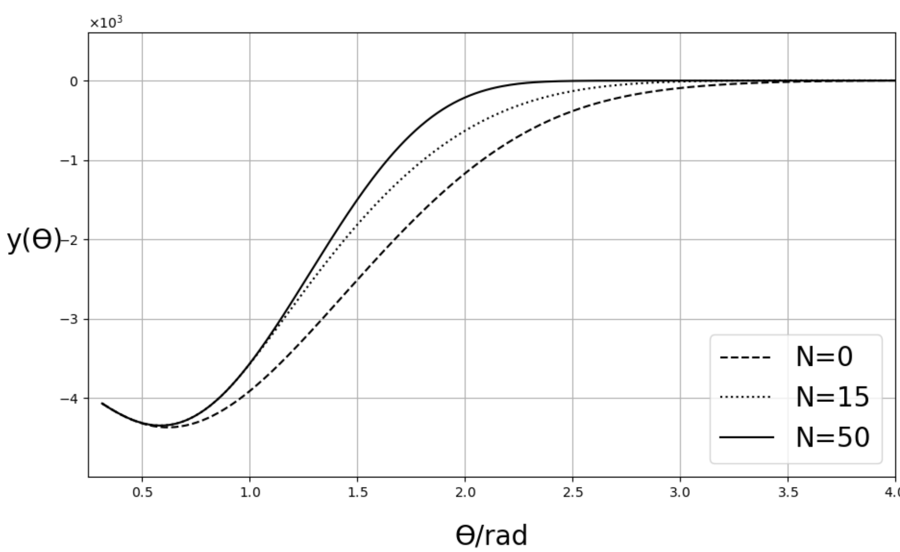

FIGURE 3. The position trajectories in y-axis direction with different switching points for N = 0, 15, 50

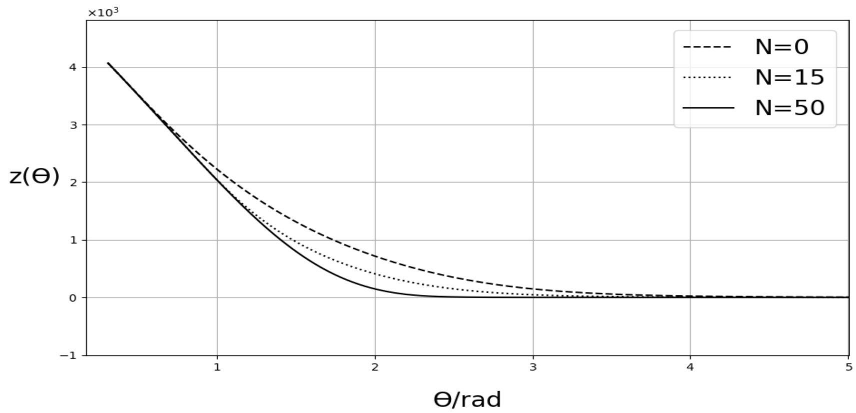

FIGURE 4. The position trajectories in z-axis direction with different switching points for N = 0, 15, 50

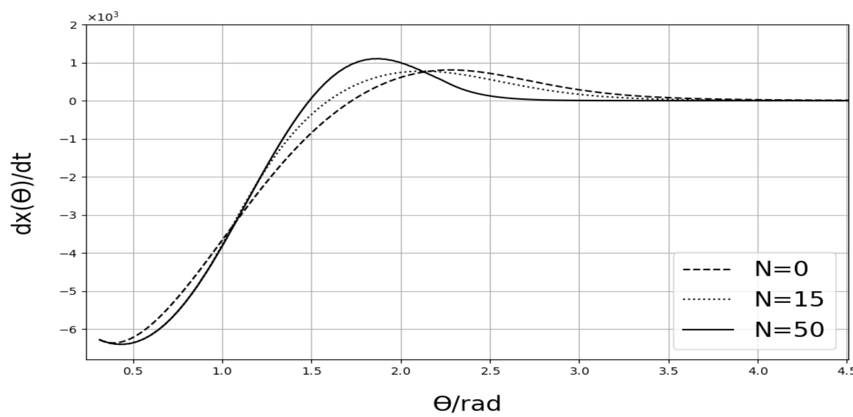

FIGURE 5. The velocity trajectories in x-axis direction with different switching points for N = 0, 15, 50

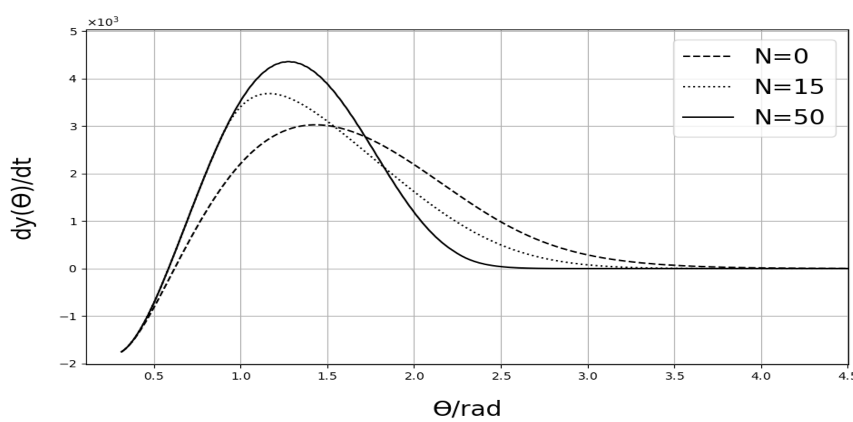

FIGURE 6. The velocity trajectories in y-axis direction with different switching points for N = 0, 15, 50

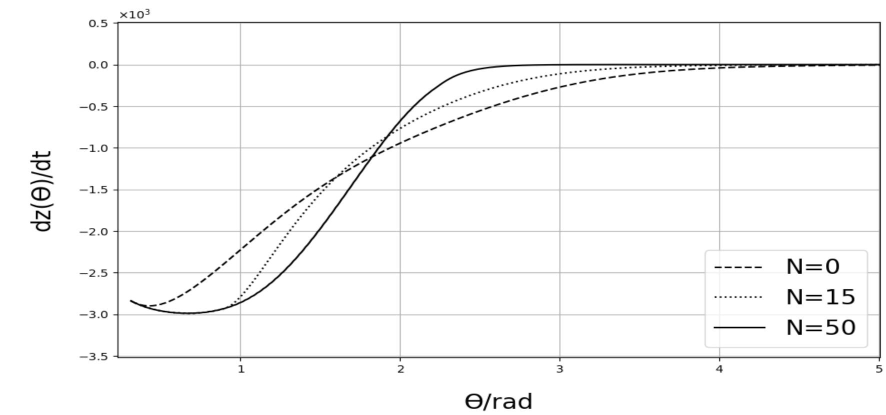

FIGURE 7. The velocity trajectories in z-axis direction with different switching points for N = 0, 15, 50

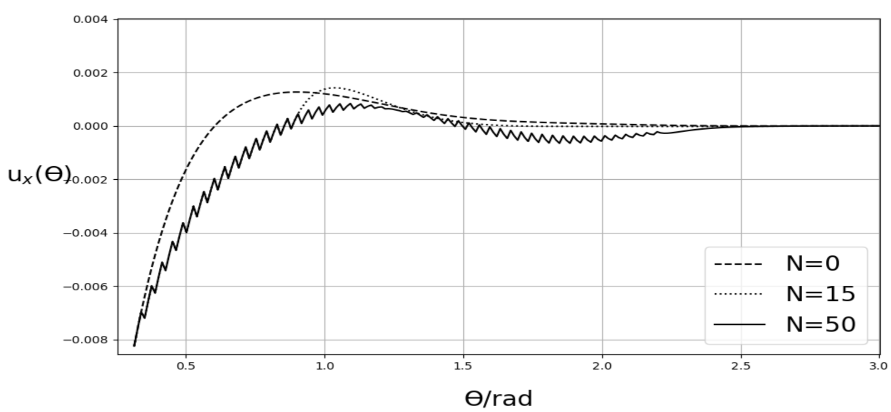

FIGURE 8. Control input signals with different switching points for N = 0, 15, 50

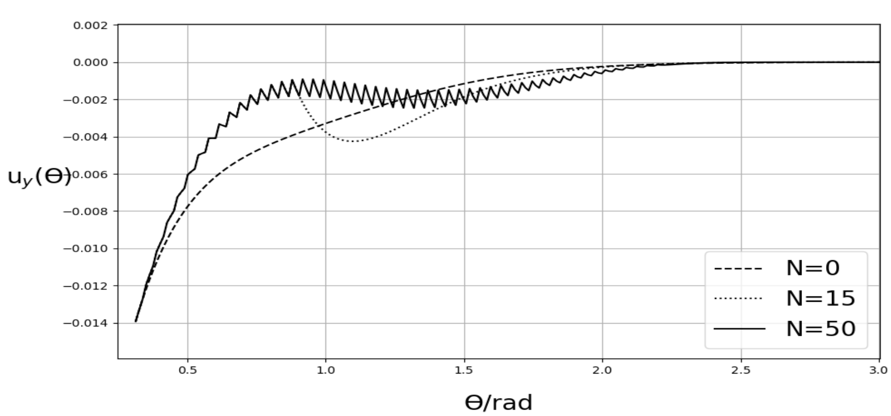

FIGURE 9. Control input signals with different switching points for N = 0, 15, 50

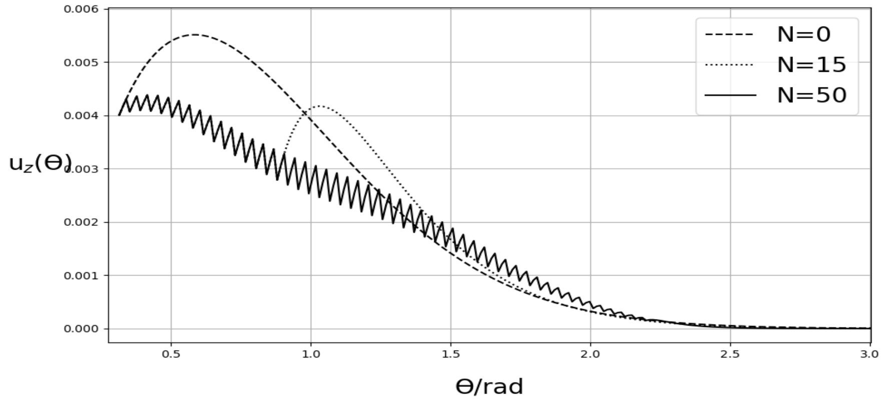

FIGURE 10. Control input signals with different switching points for N = 0, 15, 50

6. CONCLUSION

In this paper, a discrete gain scheduling control approach has been developed to solve the input saturation control problem of the elliptical orbital spacecraft rendezvous system. The main advantage of the designed control is that the system dynamic performance can be improved as the discrete dynamic parameter values gradually increase. The effectiveness of the designed gain scheduling control approach have been demonstrated through a simulation example.

APPENDIX

Step 1. Let $P(\theta,\gamma)$ is the unique positive definite solution of the PRE (4.1), then the matrix inequality $P(\theta,\gamma) \le \hat{P}(\theta,\gamma)$ holds, where $P(\theta,\gamma) = Q^\top(\theta,\gamma)\Lambda(\theta,\gamma)Q(\theta,\gamma)$, $Q(\theta,\gamma)$ is an orthogonal matrix, $\Lambda(\theta,\gamma) = \mathrm{diag}\,\{\lambda_1(\theta,\gamma),\cdots,\lambda_6(\theta,\gamma)\}$, $\lambda_i(\theta,\gamma) \ge \lambda_{i+1}(\theta,\gamma) > 0, \quad \forall i \in \mathbf{I}[1,5]$, and $\hat{P}(\theta,\gamma) = Q^\top(\theta,\gamma)\hat{\Lambda}(\gamma)Q(\theta,\gamma)$ is a positive definite matrix , whose eigenvalues are independent of $\theta \in [0,2\pi]$,

$$\hat{\Lambda}(\gamma) = \mathrm{diag}\left\{\hat{\lambda}_1(\gamma),\cdots,\hat{\lambda}_6(\gamma)\right\}, \hat{\lambda}_j(\gamma) = \max_{\theta\in[0,2\pi]} \lambda_j(\theta,\gamma), \forall j \in \mathbf{I}[1,6].$$

Step 2. Let the discrete parameter set $\Gamma_N = \{\gamma_0,\gamma_1,\cdots,\gamma_N\}, \quad \gamma_{i-1} < \gamma_i, \quad i \in \mathbf{I}[1,N]$, and let $\hat{\rho}$ be the maximum value of $\rho(\theta)$ on $[0,2\pi]$. then the ellipsoid set $O_\omega(\hat{\Lambda}_{\max}(\gamma_i)), i \in \mathbf{I}[0,N]$ satisfies the follow nested relationship: $O_\omega(\hat{\Lambda}_{\max}(\gamma_0)) \supset O_\omega(\hat{\Lambda}_{\max}(\gamma_1)) \supset \cdots \supset O_\omega(\hat{\Lambda}_{\max}(\gamma_N))$, where $O_\omega(\hat{\Lambda}_{\max}(\theta,\gamma_i)), i \in \mathbf{I}[0,N]$, will be defined later by (6.1). where

$$O_\omega(\hat{\Lambda}_{\max}(\gamma_i)) = \{\boldsymbol{\zeta} \in \mathbb{R}^{12} : \boldsymbol{\zeta}^\top \hat{\Lambda}_{\max}(\gamma_i)\boldsymbol{\zeta} \le \omega^2, \omega > 0\} \tag{6.1}$$

$$\hat{\Lambda}_{\max}(\gamma_i) = \hat{k}\mathrm{diag}\left\{\Lambda_{\max}(\gamma_i), \hat{\lambda}_1^2(\gamma_i)\lambda_{\min}^{-1}(P_e)\Lambda_e\right\}$$

$$\hat{k} = 2\hat{r}^{-2}k^8\hat{\rho}^6, \quad \Lambda_{\max}(\gamma_i) = \hat{\lambda}_1(\gamma_i)\hat{\Lambda}_(\gamma_i), \quad i \in \mathbf{I}[1,N]$$

and P_e is a positive definite matrix solution of the linear time-varying matrix inequality (2.14), Λ_e is from its orthogonal decomposition: $P_e = Q_e^\top \Lambda_e Q_e$, Q_e is an orthogonal matrix, $\Lambda_e > 0$ is a diagonal matrix.

Step 3. Let

$$\hat{P}_{\max}(\theta,\gamma_i) = \hat{Q}^\top(\theta,\gamma_i)\hat{\Lambda}_{\max}(\gamma_i)\hat{Q}(\theta,\gamma_i), i \in \mathbf{I}[1,N] \tag{6.2}$$

where $\hat{Q}(\theta,\gamma_i) = \mathrm{diag}\{Q(\theta,\gamma_i), Q_e\}, i \in \mathbf{I}[1,N]$. Define the following ellipsoids:

$$O(\hat{P}_{\max}(\theta,\gamma_i)) = \{\boldsymbol{\zeta} \in \mathbb{R}^{12} : \boldsymbol{\zeta}^\top \hat{P}_{\max}(\theta,\gamma_i)\boldsymbol{\zeta} \le \omega^2\}, \quad i \in \mathbf{I}[1,N] \tag{6.3}$$

Based on the relationship of the ellipsoid set (6.1) and (6.3), the switching points $\theta_i, i \in \mathbf{I}[1, N]$, of the discrete gain scheduling control are determined by using the rotation (reflection) transformation technique.

Statements and Declarations

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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